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Asymptotic Expansions for Two Dimensional Symmetrical Laminar Wakes

A non parallel extension of the Gaussian asymptotic representation of the two dimensional laminar incompressible far wake past a symmetrical body is presented. Under the one and only condition that the middle and far field be governed by the thin shear layer theory that keeps the complete non linearity of the equation of motion, we determined a solution in terms of an infinite power series of the streamwise space variable with fractional negative exponents. The general n-th order term has been analytically established.
The behaviour of these expansions inserted into the Navier-Stokes equations was analyzed to verify the consistency of the approximation in the intermediate region of the wake. At the third order the correction due to pressure variations identically vanishes while the contribution of the longitudinal diffusion is still two-three order of magnitude smaller than that of the transversal diffusion, depending on the Reynolds number.

Key words: asymptotic expansion, laminar, wake basic flows

MSC (2000): 76D25, 76M45

1. Introduction

For the first time determined up to the n-th order term, an asymptotic expansion is here presented as representation of the steady laminar two dimensional wake flow. Besides the far field, it is inclusive of the middle region – a useful tool as basic flow in studies concerning the role of the non parallelism on the wake instability. The position adopted is that the model of Prandtl's boundary layer with a zero pressure gradient does describe the middle and far wake field but it is unable to describe the near one. The representations are looked for the steady solutions belonging to the Reynolds number range $20 < \mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$, where $\mathrm{Re}_{\mathrm{cr}} \approx 40$ is the critical value for the onset of the first wake instability, and for the laminar steady, but unstable, solutions up to the second critical Re. The asymptotic expansion is built over all the negative powers of the longitudinal coordinate x that are integer multiple of 1/2. The coefficients describe the cross-stream distribution of momentum and are function of the lateral coordinate y transformed, according to the quasi-similarity hypothesis, to $\eta = x^{-1/2}y$. While the first order term has an exponential lateral decay the terms of order higher than the second have necessarily an algebraic decay of the kind η^{-3}. For both $x, y \to \infty$ our solution shows a lateral exponential decay as it must be because in this situation, i.e. inside the self-similar far wake, the Prandtl b.l. equations are very well approximated at any value of the control parameter by the Oseen linear form that implies: i – that $|U - u| \ll U$, ii – that the component of acceleration in the x-direction be approximated by $U\,\partial u/\partial x$, while that in the cross-stream direction be negligible (U, u are respectively the upstream velocity and the streamwise velocity component). For finite values of x, inside the middle wake, there is a fast algebraic lateral decay. In this region the Oseen approximation does not hold and the rapid decay – not rigorously demonstrable even in the far wake region, that explains why it is used as – and called – a principle, see Stewartson ([17], p. 177), Chang ([7], p. 834), Kida ([13], p. 949) – must not be expected. Going upstream this property is little by little lost due to the continuous transition from a nearly linear convection to a full non linear one.

The literature presents analytical solutions relevant to the far wake. Apart the famous Gaussian asymptotic representation by Tollmien [18], obtained after linearization of the Prandtl equations by means of the Oseen approximation, other higher order expansions, however truncated without remainder determination, were found in times past by Goldstein [10], Stewartson [17], Chang [7] and more recently by Kida [13]. However all these authors worked under the Oseen successive approximations that linearize the equation of motion while bringing it to the structure of the linear heat conduction equation in a plane solid, that, as well known, foresees asymptotic exponential decays. These expansions contain logarithmic terms to remain adherent to the Oseen approximation and thus to keep the exponential nature of the lateral decay at the higher orders of accuracy. See Stewartson ([17], cf. p. 177) that clearly describes the approach characterizing these studies and presents the chain of hypothesis used, namely: i – the difference between the linearized and the non linearized parts of the convection, called $F(x, y)$, is supposed exponentially small for large y – we demonstrated that this is not true from the third order of accuracy on (Appendices A and B) – and let play the role of inhomogeneous part of the equation of motion that in such a way becomes linear, ii – the function $u_0 = f(x_0, y)$, the initial wake profile, is the same supposed exponentially small at large y (these two assumptions are very strong because it is impossible to support them on experimental evidences, either of laboratory or numerical nature), iii – among all the terms of the solution coming from the contribution of $F(x, y)$ only the ones that dominate at large x were considered (cf. above eq. 2.10, p. 177) meaning that only the far portion of the wake is going to be described. We used none of these limitations. Our solution results an extension, however not antithetical, with regard to the expansions found by the authors above.

Our expansions contemplate the phenomenon of entrainment of ambient fluid from the sides that wakes, likewise jets, boundary and free shear layers, are experiencing as a consequence of the combination of momentum conservation and energy dissipation. The amount of fluid being transported by the wake increases downstream to match asymptotically the amount transported at infinity upstream. As we show in Appendix B, as y goes to ∞ every term of the u series (for $n > 0$) vanishes, while every term of the v series (for $n \geq 2$) goes to a constant value (with sign towards the wake center) making v vanishing as $x^{-3/2}$ when $x \to \infty$. Thus when y is large but x is finite, the streamwise component of the velocity matches the upstream velocity and the lateral component assumes a constant value that in its turn will go to zero when $x \to \infty$. This pattern agrees with the far field description in Batchelor's monography ([4], par. 5.12), where a uniform stream is superposed on a source, centered on the body generating the wake, compensating for the mass flux deficit in the wake.

A posteriori we verified the internal coherency of the model using the Navier-Stokes equations, introducing the pressure gradient and longitudinal divergence of the viscous stress. We have determined that, at worst at the fourth order of accuracy, the thin shear layer approximation could ask for corrections to be used getting close to the near wake and at small Reynolds numbers. This is so since at the third order the longitudinal diffusion term is more than two orders of magnitude lower than the leading terms and the pressure gradient vanishes identically.

This solution was applied to compute the velocity profiles in the wake of the circular cylinder, the more fundamental field among flows past a bluff body. The experimental results available in literature, up to the first critical Reynolds number for the onset of instability, are not very abundant as far wake velocity profiles are looked for and not in high mutual agreement: the more complete experiments are the ones by Kovásznay [14] and by Nishioka and Sato [15]. We compared the behaviour of our velocity profiles to the first group of results. In § 2 we present the basic hypotheses adopted. In § 3 the determination of the analytical general solution is fully described, together with the discussion of the behaviour of the expansion as $|\eta| \to \infty$ and an examination of its convergence properties. In § 4 we discuss the limits that the thin shear layer approximation imposes on these results by direct comparison with the behaviour of the present analytical procedure as applied on the Navier-Stokes equations. In § 5 the solution found for the circular cylinder wake is presented. Final comments and summary of the work are given in § 6. Details concerning the analytical procedure of integration of the equations of motion here adopted and particular properties of the expansion are illustrated in the Appendices.

2. Basic hypothesis

We assume that the two dimensional incompressible laminar stationary wake flow is described by Prandtl's adimensional boundary-layer equations

$$u\,\partial_x u + v\,\partial_y u = R^{-1}\,\partial_y^2 u\,, \tag{1}$$

$$\partial_x u + \partial_y v = 0 \tag{2}$$

where the pressure gradient, imposed by the outer flow, is equal to zero. The domain considered is composed by the intermediate and far wake

$$d < x < \infty\,, \qquad -\infty < y < \infty \tag{3}$$

where x is the standard longitudinal coordinate – with the origin placed on the centre of the body generating the wake – and the function $d = d(R) > 0$ is the distance from the center of the body beyond which the thin shear layer model becomes relevant. It is a function decreasing with R as it will be shown in the following, see § 5. Either this arbitrary origin or the near wake, that includes the symmetrical adherent vortices, fall outside the domain of validity of the equations of motion written under the thin shear layer approximation. The adimensionalization adopted is based on the characteristic length of the flow (the length D of the body generating the wake) and the velocity of the free stream U. The boundary conditions are

$$\lim_{|y|\to\infty} u(x,y) = 1\,, \qquad x \geq d\,, \tag{4}$$

$$v(x,0) = \partial_y u(x,0) = 0\,, \qquad x \geq d\,, \tag{5}$$

$$u(x_*,y) = u_*(y)\,, \qquad x_* > d\,. \tag{6}$$

The profile $u_*(y)$ is placed in the intermediate field and it is of experimental nature, either the result of a numerical simulation or of a laboratory measurement. Let us postpone until § 4 the analysis of the limits of this approximation; and introduce the virtual reference system $(x_v, y) = (x - d, y)$, whose origin is $O_v \equiv (d, 0)$ in the old reference system. It has to be determined *a fortiori* as the fictitious origin, which leads to the correct far field solution based on a self-similar first order asymptotic expansion solution. Then, we introduce the quasi-similarity transformation for the independent variables:

$$\begin{aligned} \xi &= x_v\,, \qquad && \partial_{x_v} = \partial_\xi - \tfrac{1}{2}\,\xi^{-1}\eta\,\partial_\eta\,, \\ \eta &= x_v^{-1/2}y\,, \qquad && \partial_y = \xi^{-1/2}\,\partial_\eta\,. \end{aligned} \tag{7}$$

Renaming $\xi \to x$, in what follows the virtual coordinates are (x, y) and the coordinates centered on the cylinder are $(x + d, y)$. The transformed equation of momentum becomes

$$u\,\partial_x u - \tfrac{1}{2}\,x^{-1}\eta u\,\partial_\eta u + x^{-1/2} v\,\partial_\eta u = \frac{1}{R}\,x^{-1}\,\partial_\eta^2 u\,. \tag{8}$$

The asymptotic expansion hypothesis for the velocities is now introduced:

$$\begin{aligned} u &= \sum_{n=0}^{\infty} x^{-n/2}\phi_n(\eta) = \phi_0(\eta) + x^{-1/2}\phi_1(\eta) + x^{-1}\phi_2(\eta) + \dots\,, \\ v &= \sum_{n=0}^{\infty} x^{-(n+1)/2}\chi_n(\eta) = x^{-1/2}\chi_0(\eta) + x^{-1}\chi_1(\eta) + \dots \end{aligned} \tag{9}$$

where the kind of power operating on x is determined by the condition that the total momentum per unit time across any section x must be constant (SCHLICHTING [16], p. 177). The series may converge for $x > 1$, i.e. for distances greater than $d + 1$ from the center of the body. The boundary condition (4) yields directly $\phi_0 = 1$.

The determination of the position of the virtual origin should be accomplished by adjusting the first order solution to an accurate far field numerical or experimental profile. We used numerical profiles obtained through an adaptive discretization of the stationary and incompressible Navier-Stokes equations by stabilized Finite Element Methods (BERRONE [5, 6]). For the approximated expansion solution given in § 5 the domain of the reference numerical solution reached the far field region (180 diameters behind the circular cylinder), while the global true error indicator for the reference numerical simulation, measured in the norm, was fixed at 1.5%. The relationship between χ_n and ϕ_n coming from the continuity equation is the following:

$$\chi_n = \frac{\eta}{2}\,\phi_n + \frac{n-1}{2}\,\Phi_n \tag{10}$$

where $\Phi_n(\eta)$ is an odd function defined as

$$\Phi_n(\eta) = \int_0^\eta \phi_n(\zeta)\,d\zeta\,. \tag{11}$$

Given that $\phi_0 = 1$, relation (10) yields $\chi_0 = 0$.

The general equation for ϕ_n is now obtained by substituting (9) in eq. (8) and by equating coefficients of like powers of x:

$$\mathcal{L}_n\phi_n \equiv R^{-1}\phi_n'' + \tfrac{1}{2}\,\eta\,\phi_n' + \tfrac{1}{2}\,n\,\phi_n = T_n\,, \tag{12}$$

where the primes denote the η-derivatives. Here $T_0 = T_1 = 0$, $T_2 = -\frac{1}{2}\,\phi_1^2$, and for $n \geq 3$

$$T_n = -\frac{n}{4}\sum_{i=1}^{n-1}\phi_i\phi_{n-i} + \sum_{i=1}^{n-2}\Big(-\frac{\eta}{2}\,\phi_i'\phi_{n-i} + \phi_i'\chi_{n-i}\Big) \tag{13}$$

(see Appendix A). These are linear second order ODE's and constitute an infinite succession. The linearity is due to the fact that the function $T_n(\eta)$, that keeps the same properties of smoothness of u and v, contains the products of ϕ_i, χ_i and their derivatives, where $i = 1, \dots, n-1$. The right hand side term of (12) is never a function of ϕ_n and at the same time it is the only term where the non linearity of the system is confined.

3. Determination of the solution

The solution of (12) is equal to the sum of the general solution φ_n of its homogeneous associated equation and of a particular solution $\hat{\phi}_n$.

The homogeneous equation $\mathcal{L}_n\varphi_n = 0$ associated to (12) has the solution, obtained by means of standard techniques,

$$\varphi_n = \mathrm{e}^{-(R/4)\,\eta^2}\left[C_{1n}\eta\,{}_1\mathrm{F}_1\left(1 - \frac{n}{2}, \frac{3}{2}; (R/4)\,\eta^2\right) + C_{2n}\,{}_1\mathrm{F}_1\left(\frac{1-n}{2}, \frac{1}{2}; (R/4)\,\eta^2\right)\right], \tag{14}$$

where ${}_1\mathrm{F}_1$ is the confluent hypergeometric function (see KAMKE [11], pp. 427, 473, 475). The accessory conditions require ϕ_n and φ_n even. Thus, $C_{1n} = 0$ must be imposed.

For the inhomogeneous equation

$$R^{-1}\hat{\phi}_n'' + \tfrac{1}{2}\,\eta\,\hat{\phi}_n' + \tfrac{1}{2}\,n\,\hat{\phi}_n = T_n \tag{15}$$

the determination of a particular solution is obtained through the substitution

$$\hat{\phi}_n(\eta) = t_n(\eta)\,\partial_\eta^{n-1}\gamma(\eta) \tag{16}$$

($n > 0$) where $\gamma(\eta) = \mathrm{e}^{-(R/4)\,\eta^2}$ is a transformation function. The transformed equation one gets is

$$t''_n + P_n t'_n = Q_n \,, \tag{17}$$

where the P_n are known polynomials (see Appendix A) and

$$Q_n(\eta) = R\,\frac{T_n(\eta)}{\partial_\eta^{n-1}\gamma(\eta)}\;. \tag{18}$$

Eq. (17) is solved for t', getting

$$t'_n(\eta) = \mathrm{e}^{-S_n(\eta)}\,[C_{t'} + \textstyle\int Q_n(\eta)\,\mathrm{e}^{S_n(\eta)}\,\mathrm{d}\eta] \tag{19}$$

where the function

$$S_n(\eta) = \textstyle\int P_n(\eta)\,d\eta \tag{20}$$

may be rewritten in a simple form, as shown in A (the possible singularities of the integrands are discussed at page 5).

The integration constant $C_{t'}$ can be determined by considering the asymptotic behaviour of S_n: it is easy to see that $S_n \to -\infty$ as $\eta \to \infty$ (see Appendix A). Thus, the solution (19) contains a singular term $C_{t'}\,\mathrm{e}^{-S_n(\eta)}$, that must be removed by setting $C_{t'} = 0$. Function t_n is obtained by integration of eq. (19):

$$t_n(\eta) = \textstyle\int t'_n(\eta)\,d\eta + C_t\,. \tag{21}$$

The new constant C_t can be determined reasoning on the parity of ϕ_n and successive transformations, it can be shown that the choice $C_t = 0$ leads always to a $\hat{\phi}_n$ even. Since we are considering the determination of a particular solution of eq. (12), this choice gives no loss of generality, neither spoils the correct satisfaction of the boundary condition.

Then, using eq. (16) and the relations

$$\partial_\eta^{n-1}\gamma(\eta) = (-\tfrac{1}{2})^{n-1}\,R^{(n-1)/2}\,\mathrm{Hr}_{n-1}(\eta)\,\mathrm{e}^{-(R/4)\,\eta^2}\,, \tag{22}$$

$$Q_n(\eta) = \frac{RT_n(\eta)}{\partial_\eta^{n-1}\gamma(\eta)} = A^n\,\frac{R\tilde{T}_n(\eta)}{\partial_\eta^{n-1}\gamma(\eta)} \tag{23}$$

(here $\mathrm{Hr}_n(\eta) = \mathrm{H}_n[\frac{1}{2}\,R^{1/2}\eta]$ are Hermite polynomials, constant A^n can always be factorized from T_n – see Appendix A), $\hat{\phi}_n$ can be written explicitly as

$$\hat{\phi}_n(\eta) = RA^n\,\mathrm{e}^{-(R/4)\,\eta^2}\,\mathrm{Hr}_{n-1}(\eta)\,F_n(\eta) \tag{24}$$

where

$$F_n(\eta) = \int \mathrm{e}^{-S_n(\eta)}\,G_n(\eta)\,d\eta\,, \qquad G_n(\eta) = \int \frac{\tilde{T}_n(\eta)\,\mathrm{e}^{S_n(\eta)}}{\mathrm{Hr}_{n-1}(\eta)\mathrm{e}^{-(R/4)\,\eta^2}}\,d\eta\,. \tag{25}$$

The complete solution ϕ_n is obtained by summing φ_n and $\hat{\phi}_n$:

$$\phi_n(\eta) = A^n\,\mathrm{e}^{-(R/4)\,\eta^2}\left[C_{n\,1}\mathrm{F}_1\left(\frac{1-n}{2}, \frac{1}{2}; (R/4)\,\eta^2\right) + R\,\mathrm{Hr}_{n-1}(\eta)\,F_n(\eta)\right], \tag{26}$$

where the integrals F_n and G_n, using eqs. (76), (77), may be rewritten as

$$F_n(\eta) = \int \frac{\mathrm{e}^{(R/4)\,\eta^2}}{\mathrm{Hr}_{n-1}^2(\eta)}\,G_n(\eta)\,d\eta\,, \tag{27}$$

$$G_n(\eta) = \textstyle\int \tilde{T}_n(\eta)\,\mathrm{Hr}_{n-1}(\eta)\,d\eta\,. \tag{28}$$

Formula (26), obtained by means of the transformation (16) with $n > 0$, holds also in the case $n = 0$, simply defining $\mathrm{Hr}_{-1} = 1$.

We will now consider the asymptotic behaviour as $|\eta| \to \infty$ of the solutions (26). We will see that the generic ϕ_n has necessarily an algebraic decay, except when $n = 1, 2$.

At the first order, the asymptotic expansion gives velocity profiles decaying very rapidly, because ϕ_1 is a Gaussian function (see § 3.1).

The second order solution has a more complicated behaviour. As we show in appendix B, the decay of ϕ_2 is Gaussian only when $C_2 = 0$, i.e., when the first part of ϕ_2 (involving the confluent hypergeometric function) vanishes, whilst for $C_2 \neq 0$ the decay is of the kind $\phi_2 \sim \eta^{-2}$.

The study of the asymptotic behaviour for $n \geq 3$ may be carried out by considering the properties of the confluent hypergeometric function, involved in the first part of ϕ_n, and the behaviour of the integral F_n, that dominates the second part of ϕ_n. This integral depends on the other integral G_n, and finally on the inhomogeneous term T_n, that is made up of a combination of the ϕ_i of the previous orders ($i = 1, \dots n-1$) with their derivatives, and of the χ_n. The complete analysis is presented in Appendix B. At the third order, it turns out that there are no more chances to obtain a Gaussian decay: actually, it can be shown that $\phi_3 \sim \eta^{-3}$, even if ϕ_2 has a Gaussian behaviour, as yet pointed out by GOLDSTEIN [10]. Furthermore, the algebraic decay law $\phi_n \sim \eta^{-3}$ turns out to hold as a general property for $n \geq 3$, together with the property $\chi_n \sim$ constant for $n \geq 2$ that physically means entrainment of flow from the ambient fluid (see again Appendix B).

On the contrary of what upheld by Goldstein, one may easily verify that far away from the wake center, introducing his notation where $\eta_g = (2\nu)^{-1/2}\,(Ud)^{1/2}\,yx^{-1/2} = (R/2)^{1/2}\,yx^{-1/2}$ (ν is the viscosity and d the reference length), the limit of $\phi_{ng} \sim \eta_g^{-3}$ as $\nu \to 0$ is zero. Thus, this would not be a reason to 'a priori' discard an asymptotic behaviour of the kind $\sim \eta^{-3}$ as he did.

According to this solution behaviour, the asymptotic form of the Prandtl equations here used may be evaluated. The result is that at $x = \text{const}$, as $\eta \to \infty$, the lateral diffusion ($\sim\eta^{-4}$) becomes progressively smaller than the convection ($\sim\eta^{-2}$). Thus asymptotically the Prandtl model moves naturally to a model of simple convective transport.

At this point, we know that solution (26) is not diverging as $|\eta| \to \infty$, but we still need to discuss the other possible singularities in the domain $|\eta| < \infty$. Considering eq. (28), the integrand is made up by the factor $\tilde{T}_n = T_n/A^n$, i.e. the inhomogeneous term of eq. (12), that is a regular function for finite η, and the Hermite polynomial Hr_{n-1}. Thus, G_n has no singularities for $|\eta| < \infty$. The integrand of eq. (27) involves instead the factor Hr_{n-1}^{-2}, that has $n-1$ poles of order 2 at the points $\{\eta_0, \eta_1, \ldots, \eta_{n-1}\}$, which are the zeroes of the Hermite polynomial. By integrating between separate singularities, F_n results a function with $n-1$ poles of order 1 at the isolated points $\{\eta_0, \eta_1, \ldots, \eta_{n-1}\}$. Now, in relation (26) the function F_n is multiplied by Hr_{n-1}, i.e. by the polynomial with $n-1$ simple zeroes at the same points $\{\eta_0, \eta_1, \ldots, \eta_{n-1}\}$, thus these singularities are eliminated.

As regard the convergence, the series representations (9) used here are asymptotic power series for u and v as $x \to \infty$ and for every value of the variable η. This is true since the set $\{x^{-n/2}\}$ ($n = 0, 1, 2, \ldots$) is an asymptotic sequence, i.e. $x^{-(n+1)/2} = o(x^{-n/2})$ for each n (definition in the sense given by Carleman, Borel). Thus, by the theorem of summation of asymptotic series (ERDÉLYI [8], ch. I) the series possesses a sum, in the generalized sense of the existence of an equivalence relation. This means that it exists a class of functions which are asymptotically equal to u as $x \to \infty$. The determination of the convergence in the ordinary sense is a very arduous problem, but it is possible to obtain partial results relevant to the part of the series corresponding to the homogeneous general solution: actually, it can be shown that this part has an ordinary sum, with a certain convergence radius, by means of the comparison criterion, provided that the succession of constants $\{C_n\}$ is bounded.

The solutions obtained have a general physical meaning, being related to a generic two dimensional wake. The procedure might be in principle extended to jets (getting again an infinite succession of linear equation for the factor functions ϕ_n, except the first) and proper boundary-layers (in this last case the hierarchy obtained is constituted by non linear ordinary differential equations) by using the relevant accessory conditions and powers series expansion in x (see SCHLICHTING [16], p. 734). Note that, up to this point, only the conditions at $x \to \infty$ and $|y| = 0$, $|y| \to \infty$ were used. The constant A and $\{C_n\}$ are to be determined through the boundary condition at $x = x_*$, which are specified by an experimental velocity profile.

3.1 Solutions of the first four order equations

Using relations (26) and (10) the following results may be easily obtained:

Order 0 $[O(x^{-1})]$.

equation: $$\frac{1}{R}\,\phi_0'' + \frac{\eta}{2}\,\phi_0' = 0\,, \tag{29}$$

solution: $$\phi_0 = A^0\,\mathrm{e}^{-(R/4)\,\eta^2}\,C_0\,{}_1\mathrm{F}_1(\tfrac{1}{2}, \tfrac{1}{2}; (R/4)\,\eta^2) = A^0 C_0 = 1\,, \tag{30}$$

$$\chi_0 = -\frac{1}{2}\,\Phi_0 + \frac{\eta}{2}\,\phi_0 = -\frac{1}{2}\int d\eta + \frac{\eta}{2} = 0\,. \tag{31}$$

The property ${}_1\mathrm{F}_1(a, a; z) = \mathrm{e}^z$ has been applied. Constant $C_0 = 1$ is determined directly through the accessory condition at $x \to \infty$.

Order 1 $[O(x^{-3/2})]$.

equation: $$\frac{1}{R}\,\phi_1'' + \frac{\eta}{2}\,\phi_1' + \frac{1}{2}\,\phi_1 = 0\,, \tag{32}$$

solution: $$\phi_1 = A\,\mathrm{e}^{-(R/4)\,\eta^2}\,C_1\,{}_1\mathrm{F}_1(0, \tfrac{1}{2}; (R/4)\,\eta^2) = AC_1\,\mathrm{e}^{-(R/4)\,\eta^2} = -A\,\mathrm{e}^{-(R/4)\,\eta^2}\,, \tag{33}$$

$$\chi_1 = \frac{\eta}{2}\,\phi_1 = -\frac{A}{2}\,\eta\,\mathrm{e}^{-(R/4)\,\eta^2}\,. \tag{34}$$

The property ${}_1\mathrm{F}_1(0, a; z) = 1$ was used. Constant AC_1 should be determined by the accessory conditions of eq. (1); it must be negative because ϕ_1 represent a velocity defect. It is possible to choose $C_1 = -1$ to get a value $A > 0$, which should be related to the drag coefficient (see § 5). At the first order the well-known asymptotic Gaussian solution behaviour was recovered (SCHLICHTING [16], BATCHELOR [4]).

Order 2 $[O(x^{-2})]$.

$$\text{equation:}\quad \frac{1}{R}\,\phi_2'' + \frac{\eta}{2}\,\phi_2' + \phi_2 = -\frac{1}{2}\,A^2\,\mathrm{e}^{-(R/2)\,\eta^2}\,, \tag{35}$$

$$\begin{aligned}\text{solution:}\quad \phi_2 &= A^2\,\mathrm{e}^{-(R/4)\,\eta^2}\,\left[C_2\,{}_1\mathrm{F}_1(-\tfrac{1}{2},\tfrac{1}{2};(R/4)\,\eta^2) + R\,\mathrm{Hr}_1(\eta)\,F_2(\eta)\right]\\ &= A^2\,\mathrm{e}^{-(R/4)\,\eta^2}\left[C_2\,{}_1\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{2};(R/4)\,\eta^2\right) + \mathrm{e}^{-(R/4)\,\eta^2} + \frac{1}{2}\,\sqrt{\pi R}\;\eta\,\mathrm{erf}\left(\frac{\sqrt{R}}{2}\eta\right)\right],\end{aligned} \tag{36}$$

$$\begin{aligned}\chi_2 = \frac{A^2}{2}\,&\left\{C_2\left[\int \mathrm{e}^{-(R/4)\,\eta^2}\,{}_1\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{2};(R/4)\,\eta^2\right)d\eta + \frac{1}{2}\,\eta\,\mathrm{e}^{-(R/4)\,\eta^2}\,{}_1\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{2};(R/4)\,\eta^2\right)\right]\right.\\ &\left.+\frac{1}{2}\,\eta\,\mathrm{e}^{-(R/2)\,\eta^2} + \sqrt{\frac{\pi}{2R}}\;\mathrm{erf}\left(\frac{\sqrt{R}}{2}\,\eta\right) - \left(\sqrt{\frac{\pi}{R}} - \frac{\sqrt{\pi R}}{4}\,\eta^2\right)\mathrm{e}^{-(R/4)\,\eta^2}\;\mathrm{erf}\left(\frac{\sqrt{R}}{2}\eta\right)\right\}.\end{aligned} \tag{37}$$

Here the integral $F_2(\eta)$ has been calculated explicitly, and C_2 should be determined by the accessory conditions. The integral in (37) may be determined by numerical integration or approximated with rational and special functions; nevertheless.

Order 3 $[O(x^{-5/2})]$.

$$\text{equation:}\quad \frac{1}{R}\,\phi_3'' + \frac{\eta}{2}\,\phi_3' + \frac{3}{2}\,\phi_3 = -\frac{3}{2}\,\phi_1\phi_2 - \frac{1}{2}\,\eta(\phi_1\phi_2)' + \phi_2'\chi_1 + \phi_1'\chi_2\,, \tag{38}$$

$$\begin{aligned}\text{solution:}\quad \phi_3 &= A^3\,\mathrm{e}^{-(R/4)\,\eta^2}\,\left[C_3\,{}_1\mathrm{F}_1(-1,\tfrac{1}{2};(R/4)\,\eta^2) + R\,\mathrm{Hr}_2(\eta)\,F_3(\eta)\right]\\ &= A^3\,\mathrm{e}^{-(R/4)\,\eta^2}\,[C_3(1-(R/2)\,\eta^2) + R(R\eta^2-2)\,F_3(\eta)]\\ &= A^3\,\mathrm{e}^{-(R/4)\eta^2}\,(2-R\eta^2)\,[\tfrac{1}{2}\,C_3 - RF_3(\eta)]\,.\end{aligned} \tag{39}$$

Here the presence of the common factor $(R\eta^2-2)$ is due to the fact that the Hermite polynomial Hr_2 is proportional to the confluent hypergeometric function (${}_1\mathrm{F}_1(a,b,z)$ becomes a polynomial when a is a negative integer). The function $F_3(\eta)$ may be determined by numerical integration or approximated by special and rational functions. Finally, C_3 is determined by the accessory conditions.

3.2 The wake width

The obtained expansions allow to write the dependence on x and R of the wake width δ as defined by the displacement thickness of the boundary-layer theory:

$$\delta = \frac{2}{1-u_c}\int_0^\infty (1-u)\,dy\,, \tag{40}$$

where u_c is the velocity at the wake center.

At the order 0, the wake flow is $u_0 = 1$, so eq. (40) gives $\delta = 0$.

At the first order, the wake flow is $u_1 = 1 + x^{-1/2}\phi_1(\eta) = 1 - Ax^{-1/2}\,\mathrm{e}^{-(R/4)\,\eta^2}$, and $u_c = 1 - Ax^{-1/2}$. Relation (40) gives

$$\delta = \frac{2}{Ax^{-1/2}}\int_0^\infty A\,\mathrm{e}^{-(R/4)\,\eta^2}\,d\eta = 2\left(\frac{\pi}{R}\right)x^{1/2}\,, \tag{41}$$

and we find the well-known asymptotic property $\delta \propto x^{1/2}$.

At the second order, the wake flow is $u_2 = 1 + x^{-1/2}\phi_1(\eta) + x^{-1}\phi_2(\eta)$, using one obtains

$$\delta = 2(\pi/R)^{1/2}\,x^{1/2} + A[I_{\phi_2}(C_2) - (\pi/R)^{1/2}\,(2C_2+1)] \tag{42}$$

where

$$\begin{aligned}I_{\phi_2}(C_2) &= \int_0^\infty \mathrm{e}^{-(R/4)\,\eta^2}\,[2C_2\,{}_1\mathrm{F}_1(-\tfrac{1}{2},\tfrac{1}{2};(R/4)\,\eta^2) + \mathrm{e}^{-(R/4)\,\eta^2} + \tfrac{1}{2}\,(\pi R)^{1/2}\,\eta\;\;\mathrm{erf}\left(\tfrac{1}{2}\,R^{1/2}\eta\right)]\,d\eta\\ &= 2C_2\int_0^\infty \mathrm{e}^{-(R/4)\,\eta^2}\,{}_1\mathrm{F}_1(-\tfrac{1}{2},\tfrac{1}{2};(R/4)\,\eta^2)\,d\eta + (2\pi/R)^{1/2}\,.\end{aligned} \tag{43}$$

The integral at the right hand side is equal to zero, thus the wake width is now corrected by a constant term independent of x:

$$\delta = (\pi/R)^{1/2}\,[2x^{1/2} + A(2^{1/2} - 1 - 2C_2)]\,. \tag{44}$$

By considering higher orders corrections, the wake width is found to be represented by an expansion of the kind

$$\delta(x, R) = \sum_{i=0}^{\infty} x^{1-i/2} \delta_i(R) \tag{45}$$

with $\delta_0 = 0,\ \delta_1 = 2(\pi/R)^{1/2},\ \delta_2 = (\pi/R)^{1/2}\, A(2^{1/2} - 1 - 2C_2),\ \ldots$.

4. Limits of the thin shear layer model

The asymptotic expansions we determined are solutions of the Prandtl boundary-layer equations (1), (2), deduced from the Navier-Stokes equation under the condition

$$\lambda = \frac{\delta(x, R)}{L(x)} \ll 1\ , \tag{46}$$

where the thickness δ is the internal scale of the flow and the length L is the external longitudinal scale. We checked this assumption *a posteriori* by using the expression for $\delta(x, R)$ up to the third order. Function λ is the parameter fixing the quality of the physical assumption adopted, see Fig. 1, where the maps $\lambda(x, R) = \text{constant}$ are shown for the cylinder wake, that will be described in the following paragraph. In this case we assumed an external scale equal to the distance from the cylinder center, i.e. $L(x) = x + d$.

A more rigorous analysis of the boundary-layer model validity can be made by considering the complete 2D steady Navier-Stokes equations and reasoning on the results they give once solutions in the form (9) are admitted. Writing the dependence on x of the velocity variables and their derivatives:

$$u = 1 + x^{-1/2}\phi_1(\eta) + \ldots = 1 + O(x^{-1/2})\,, \tag{47}$$

$$v = x^{-1}\chi_1(\eta) + \ldots = O(x^{-1})\,, \tag{48}$$

$$\partial_x \to \partial_x - \tfrac{1}{2} x^{-1}\eta\partial_\eta = O(x^{-1})\,, \tag{49}$$

$$\partial_y \to x^{-1/2}\partial_\eta = O(x^{-1/2})\,, \tag{50}$$

it is possible to determine the order of magnitude of each term in the Navier-Stokes equations.

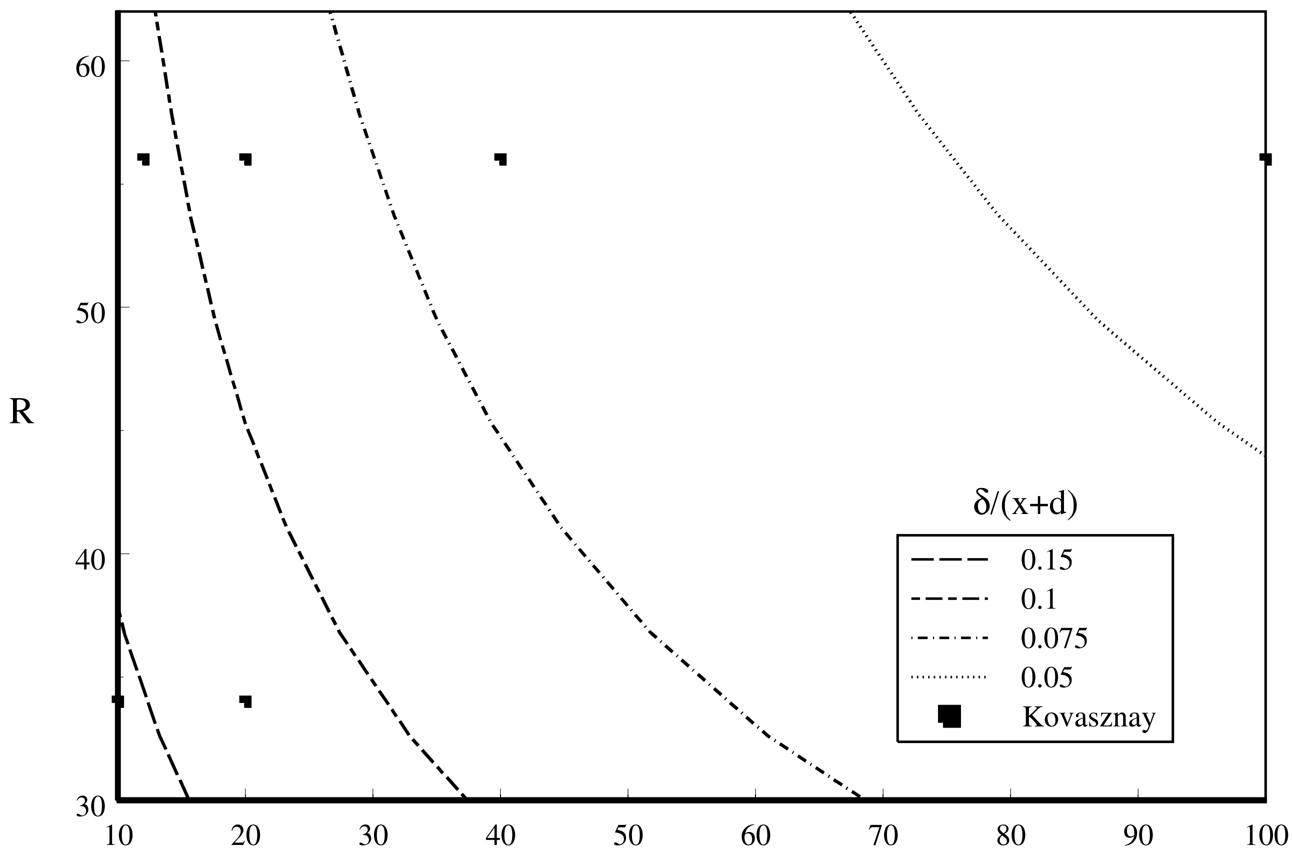

Fig. 1. Map of $\lambda(x, R) = \delta(x, R)/(x + d) = \text{constant}$. The positions of experiments by Kovásznay are shown. The abscissa starts at 10, a value typically greater than $d + 1$ (the point before which the series cannot converge) in the Reynolds numbers range considered

Since the pressure gradient $\partial_y p$ may be at most supposed of order x^{-2}, the pressure variation across the wake should be of order $x^{-3/2}$, and the general law for the pressure be of the kind $p = p_0 + x^{-3/2} p_3(\eta) + O(x^{-2})$, with $p_0 = p_\infty$. Then, the magnitude of the downstream gradient would be at most: $\partial_x p \sim x^{-5/2}$.

By substituting in the momentum balance in the x-direction the pressure law $p = p_0 + x^{-3/2} p_3(\eta) + \ldots$, together with the expressions $u = 1 + x^{-1/2}\phi_1(\eta) + \ldots$ and $v = x^{-1}\chi_1(\eta) + \ldots$, the ODEs remain the same as (12) up to the equation at $n = 3$ that becomes

$$x^{-5/2}[R^{-1}\phi_3'' + \tfrac{1}{2}\eta\phi_3' + \tfrac{3}{2}\,\phi_3 = T_3 - \tfrac{1}{2}(3p_3 + \eta p_3') + (4R)^{-1}\ (3\phi_1 + 5\eta\phi_1' + \eta^2\phi_1'')] .$$

In general, every equation at $n \geq 3$ will change:

$$x^{-(n+2)/2}[R^{-1}\phi_n'' + \tfrac{1}{2}\eta\phi_n' + \tfrac{1}{2}n\phi_n = T_n + (\text{pr. gradient terms}) + R^{-1}(\text{streamwise diffusion})] , \tag{51}$$

where the longitudinal viscous stress divergence term are known from the previous orders, and the pressure gradient terms are to be determined from the balance in the y-direction.

The present theory for the determination of the velocity profiles could be improved by simply including, after the second order has been attained, some extra terms in the inhomogeneous part of each equation $\mathcal{L}_n\phi_n = T_n$. However, it is important to check whether these corrections are really significant.

For $n = 3$, the order of magnitude of the corrections may be estimated in the following way. At the leading order, the Navier-Stokes equation for the y-direction in the coordinate system $\{x, \eta\}$ is

$$x^{-1/2}p' = R^{-1}x^{-1}v'' - u\,\partial_x v + \tfrac{1}{2}x^{-1}\eta u v' + O(x^{-5/2}) . \tag{52}$$

By substituting $u = 1 + x^{-1/2}\phi_1(\eta) + \ldots$ and $v = x^{-1}\chi_1(\eta) + \ldots$, we obtain

$$p = p_0 + x^{-3/2}p_3(\eta) + O(x^{-2}) \tag{53}$$

where

$$p_3 = R^{-1}\chi_1' + \tfrac{1}{2}\eta\,\chi_1 + \tfrac{1}{2}\int \chi_1\, d\eta\,; \tag{54}$$

by substituting the known expression $\chi_1 = -(A/2)\,\eta\,\mathrm{e}^{-(R/4)\,\eta^2}$ the result is

$$p_3 = 0\,, \tag{55}$$

hence the pressure gradient could be inserted at most in the next order equation, and we must state $p = p_0 + x^{-2}p_4(\eta) + O(x^{-5/2})$.

As regards the longitudinal divergence of the viscous stress τ_x, an easy check of its magnitude order may be directly done. Function ϕ_1 is known and thus we have

$$(4R)^{-1}\,(3\phi_1 + 5\eta\phi_1' + \eta^2\phi_1'') = -A(4R)^{-1}\,\mathrm{e}^{-(R/4)\,\eta^2}\,(\tfrac{1}{4}R^2\eta^4 - 3\eta^2 + 3)\,. \tag{56}$$

Comparison of this function with T_3 shows that it represent a correction of at most 0.6% at $R = 40$ and 0.1% at $R = 100$.

At the third order of accuracy there is no contribution from the pressure variations and the longitudinal diffusion is still two-three order of magnitude lower than: i) the lateral diffusion $(1/R)\,\phi_n''$ and ii) the non linear terms, contained in T_n. Note that the mathematical structure of eqs. (51) is not different from that of eq. (12). The above corrections simply add each one a term to the inhomogeneous part of the equation. The linearity of eqs. (2.18) is not spoiled, since the boundary-layer equations already contain the full non linearity of the Navier-Stokes equations. As a consequence, the determination of the solution for eqs. (51) would not be affected by further difficulties. Since the correction pertinent to the streamwise diffusion is so small at the third order, it is not certain which higher stage of accuracy has to be reached to get them of the same order of the lateral diffusion.

5. An approximated solution for the cylinder wake

In this section we consider the wake generated by the circular cylinder. To determine the laminar steady solution we must introduce the field information equivalent to the boundary condition at $x = x_0$, that is the velocity distributions at a number of diameters downstream of the body sufficient to be inside the part of the field where the thin shear layer hypothesis holds. Constant A can be determined through the adimensional drag coefficient

$$\frac{D}{ld\rho U^2} = \tfrac{1}{2}\,c_D \tag{57}$$

(D/l is the drag force per unit length, d the diameter, U the free stream velocity, and c_D the adimensional drag coefficient) that is equal to the total momentum loss induced in the flow by the presence of the body. Thus

$$\tfrac{1}{2}\,c_D = \int_{-\infty}^{\infty} u(1-u)\,dy\ . \tag{58}$$

where $u = u(y)$ for $x \geq x_0$. At the first order the wake flow is $u_1 = 1 + x^{-1/2}\phi_1(\eta) = 1 - Ax^{-1/2}\,\mathrm{e}^{-(R/4)\,\eta^2}$, having determined that $C_0 = 1$ through the boundary condition at $x \to \infty$ and placed $C_1 = -1$ in order to get a positive A in correspondence to a positive c_d. Substituting in eq. (58) we get

$$\tfrac{1}{2}\,c_D = -\int_{-\infty}^{\infty} [1 + x^{-1/2}\phi_1(\eta)]\,x^{-1/2}\phi_1(\eta)\,dy = -\int_{-\infty}^{\infty} x^{-1/2}\phi_1(\eta)\,dy + O(x^{-1})\;,$$

i.e.

$$\tfrac{1}{2}\,c_D = A\int_{-\infty}^{\infty} \mathrm{e}^{-(R/4)\,\eta^2}\,d\eta = 2\left(\frac{\pi}{R}\right)^{1/2} A\,. \tag{59}$$

Constant A is a function of R also because c_d, an experimentally well-known quantity, is a strong function of the Reynolds number:

$$A(R) = \tfrac{1}{4}\,(R/\pi)^{1/2}\,c_D(R)\,. \tag{60}$$

At the second order, the wake flow is $1 + x^{-1/2}\phi_1(\eta) + x^{-1}\phi_2(\eta)$, and substituting again in eq. we obtain, excluding higher order terms,

$$\tfrac{1}{2}\,c_D = -\int_{-\infty}^{\infty} \phi_1(\eta)\,d\eta - x^{-1/2}\int_{-\infty}^{\infty} [\phi_2(\eta) + \phi_1^2(\eta)]\,d\eta\,. \tag{61}$$

By separating equal powers of x:

$$\tfrac{1}{2}\,c_D = -\int_{-\infty}^{\infty} \phi_1(\eta)\,d\eta\;, \tag{62}$$

$$0 = \int_{-\infty}^{\infty} [\phi_2(\eta) + \phi_1^2(\eta)]\,d\eta\;. \tag{63}$$

The second relation gives a new condition:

$$0 = \int_{-\infty}^{\infty} [\phi_2(\eta) + \phi_1^2(\eta)]\,d\eta = A^2 C_2 \left[\int_{-\infty}^{\infty} \mathrm{e}^{-(R/4)\,\eta^2}\,{}_1\mathrm{F}_1\left(-\frac{1}{2}, \frac{1}{2}; (R/4)\,\eta^2\right) d\eta \right.$$
$$\left. + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-(R/2)\,\eta^2}\,d\eta - \frac{1}{4}\left(\frac{\pi}{R}\right)^{1/2}\int_{-\infty}^{\infty} \eta\,\mathrm{e}^{-(R/2)\,\eta^2}\,\mathrm{erf}\left(\frac{1}{2}\,R^{1/2}\eta\right) d\eta\right], \tag{64}$$

that however does not allow to determine constant C_2, since on the right hand side the first integral is zero, while the sum of the second and third integral vanishes exactly. To recover the values of the constant factors at the higher orders one has to fit an experimental distributions at $x = x_0$ inside the domain of validity of the boundary layer approximation, see also Stewartson [17] for a neat explanation of the influence such boundary information has on the factors of every term in the expansion. We used the experimental profiles by Kovásznay [14]. These are still one of the more detailed source in the literature and comprehend cross-section profiles in the middle wake at two different Reynolds numbers (steady flow at $R = 34$ and averaged unsteady flow at $R = 56$). Vice versa there is a lack of profiles relevant to the middle portion of the wake and coming from numerical solution of the Navier-Stokes equations. At $R = 34$ there are two near wake half profiles – located two and five diameters downstream where also Kovászany performed measurements – presented by Yang and Zebib [20] in a paper dealing with the first instability of the cylinder wake. But these profiles are inside a wake region where our assumption does not hold. An analogous situation prevents to contrast our profiles to other numerical results, see for instance Apelt [2], Keller and Takami [12], Nishioka and Sato [15], and Fornberg [9].

It is very recent the availability of numerical approximated profiles (Berrone [5, 6]) in the wake of the circular cylinder obtained by an *a posteriori* control error procedure based on a residual estimate, see Babuška and Rheinboldt [3], applied to an adaptive stabilized finite element discretization of the stationary incompressible Navier-Stokes equations (Verfürth [19]). We contrasted our expansion solution with a numerical solution (Berrone [5]) based on this adaptive discretization (where h is the maximum diameter of the finite element distribution) on a domain Ω that reaches 30 diameters ahead of the cylinder, 180 diameters behind it, and 60 diameters laterally. The boundary conditions were of no slip on the cylinder and of uniformity at infinity. The number of grid points was 40926 and the true error estimate, measured in the norm,

$$\frac{|\mathbf{u} - \mathbf{u_h}|_{1,\,\Omega} + \|p - p_h\|_{0,\,\Omega}}{|\mathbf{u_h}|_{1,\,\Omega} + \|p_h\|_{0,\,\Omega}} \approx 0.0149\,,$$

where Ω represents the computation domain.

We also used Berrone's data to determine the position of the virtual origin of the wake flow, relatively to the origin of the arbitrary reference system. We did it by contrasting our expansions with a numerical profile, assumed as our experimental reference far field representation, computed at 100 diameters downstream of the cylinder center. For instance at $R = 34$, the virtual origin turned out to be placed $5.1 \pm 2D$ downstream of the cylinder center. The accuracy was estimated by using the data relevant to another numerical profile placed 60 diameters downstream and at the same Reynolds number.

Writing the expression of the basic flow up to the fourth order, the constants C_2, C_3, C_4 and the virtual origin d were fitted to Kovásznay's experimental and Berrone's numerical data through the least square method. In the range $30 < R < 60$, the polynomial approximations of the factors A (given by (60)) $C_i, i = 0, 4$ and d as functions of R turned out to be

$$A(R) \approx 1.16 + 5.1 \cdot 10^{-3} R - 1.9 \cdot 10^{-5} R^2 \,, \tag{65}$$

$$C_0 = 1 \,, \tag{66}$$

$$C_1 = -1 \,, \tag{67}$$

$$C_2(R) \approx -13.0 + 0.669R - 0.0105R^2 + 5.14 \cdot 10^{-5} R^3 \,, \tag{68}$$

$$C_3(R) \approx -21. + 1.0R - 0.016R^2 + 7.9 \cdot 10^{-5} R^3 \,, \tag{69}$$

$$C_4(R) \approx -0.27 + 0.16R - 4.0 \cdot 10^{-3} R^2 + 2.5 \cdot 10^{-5} R^3 \,, \tag{70}$$

$$d \approx 11.1 - 0.177R \,. \tag{71}$$

The relative comparison of the first four orders of accuracy for these expansion solutions is shown in Figs. 2 and 3. In Fig. 4 at $R = 34$ our longitudinal velocity component distributions are in good accordance with the experimental data by Berrone and Kovásznay. In Fig. 5 at $R = 56$, a value of the control parameter where the stationary wake solution is unstable, our longitudinal velocity component distributions agree with the time averaged experimental distribution by Kovásznay. The profiles by Kovásznay are affected by a slight asymmetry associated to the hot wire probe lateral

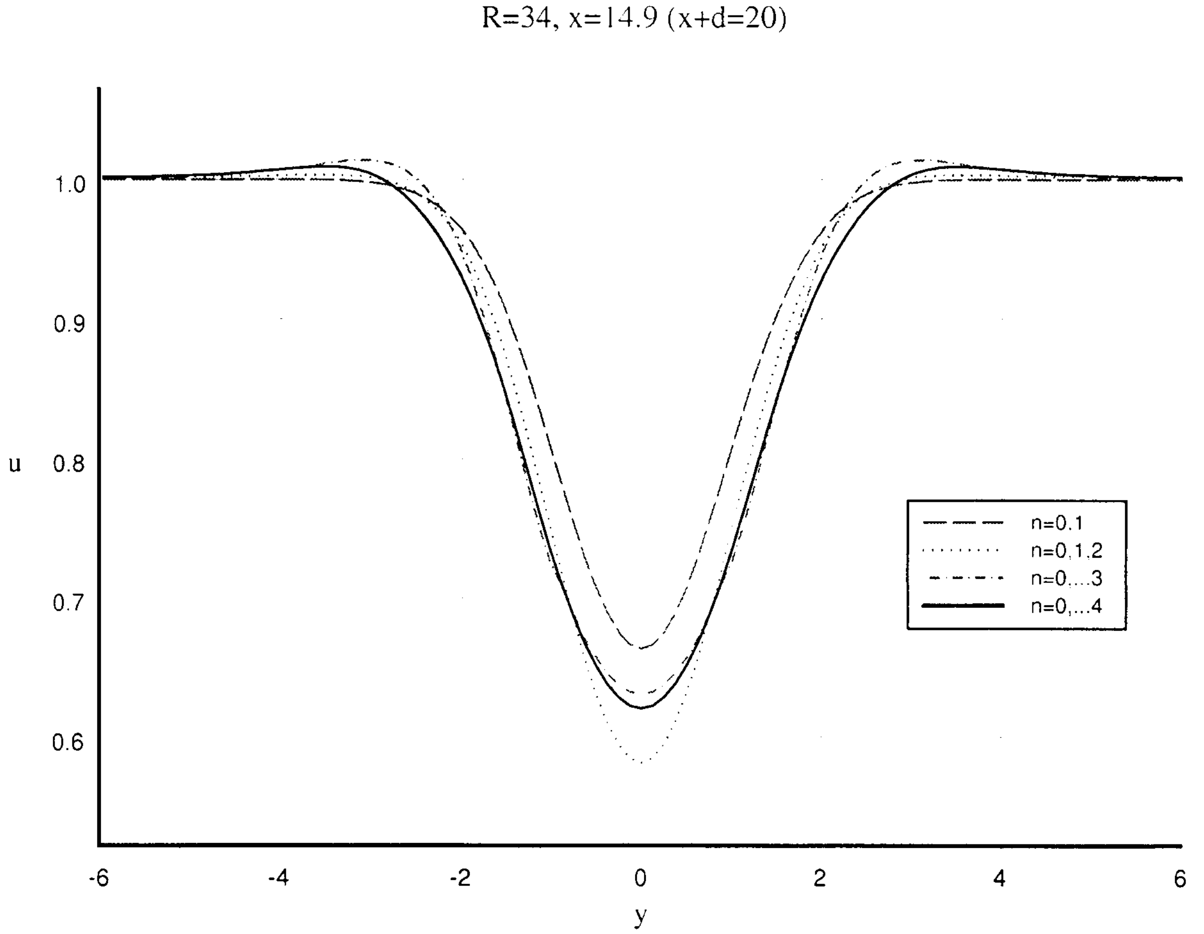

Fig. 2. Velocity component u: successive orders approximations at $R = 34$ and twenty diameters downstream of the center of the cylinder ($d = 5.1$)

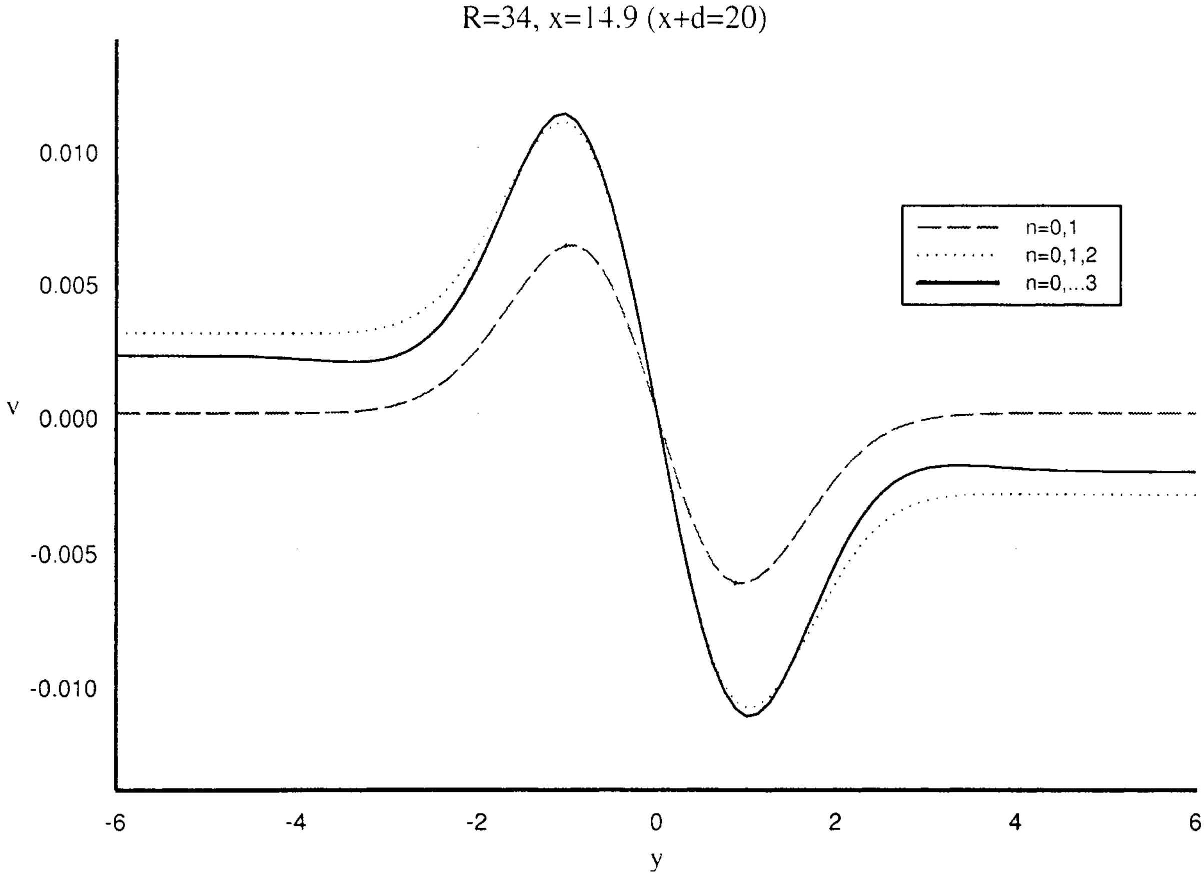

Fig. 3. Velocity component v: successive orders approximations at $R = 34$ and twenty diameters downstream of the center of the cylinder ($d = 5.1$)

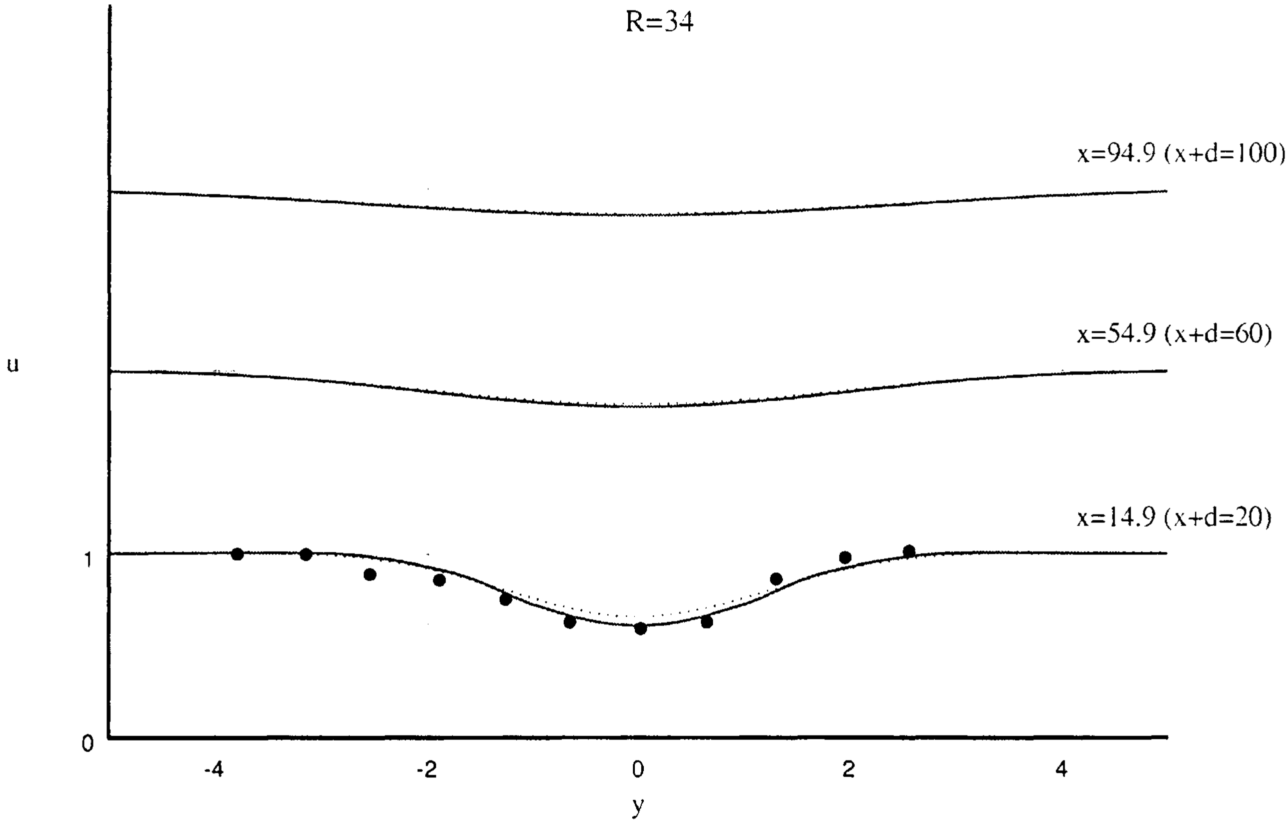

Fig. 4. $R = 34$. Velocity component u. Present expansion solution (———), Berrone's numerical solution (··········), experimental data by Kovásznay [14] (●)

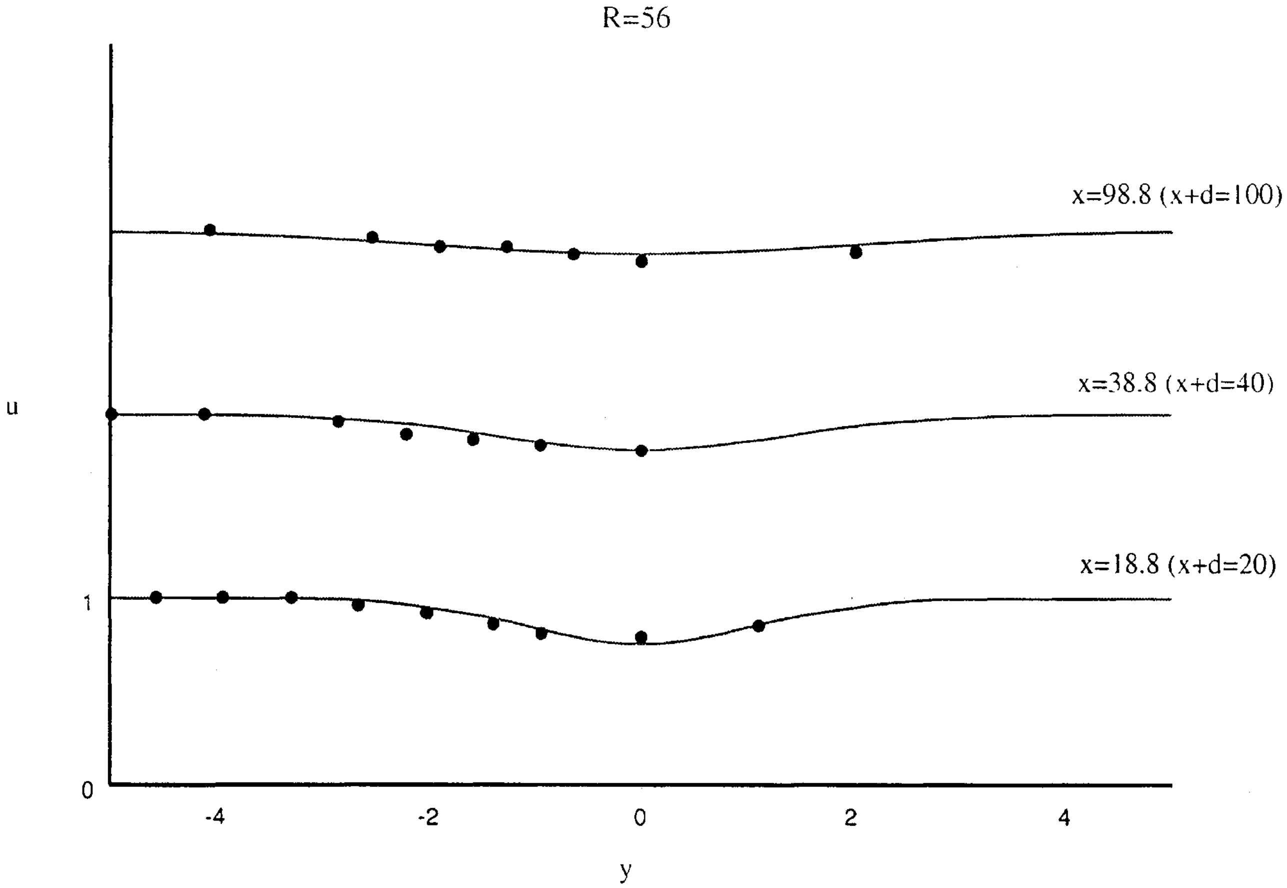

Fig. 5. $R = 56$. Velocity component u. Present expansion solution ($d = \ldots$) (———), experimental data by Kovásznay [14] (●)

placing. According to Kovásznay we assumed that the half-distribution measured before the wake center was reached be unaffected. To figure out an accuracy estimate for our results we must however estimate also the accuracy of the experiment. To this end we used the numerical simulations by Berrone as reference. At $R = 34$ a deviation $\Delta_B = \|u - u_B\|^2_{0,\mathcal{L}} / \|1 - u_B\|^2_{0,\mathcal{L}} \simeq 6.5\%$ turns out for Kovásznay's data, where u is the measured longitudinal velocity component, the transversal component being not measured, and $\mathcal{L}$ is the station at $x = 20$ diameters from the center of the cylinder. As regard our approximate solution we find a Δ_B of about 1.7%, a lower value with regard to the measurements.

6. Summary and conclusions

Solutions in the form of asymptotic expansions were determined for the laminar steady configuration of the two dimensional symmetric incompressible wake. The equations of motion are written under the thin shear layer approximation and a quasi-similarity transformation for the independent coordinates is used. The solution validity is therefore limited to the middle and far portions of the field. The expansions are power series of the streamwise spatial variable with negative fractional exponents. The power law has been selected by imposing the constancy of the total rate of momentum transport in the downstream direction. The expression of the general term of order n of the expansion was analytically obtained. Further limiting assumptions as Oseen's successive approximations or the shape of the lateral asymptotics were not used.

The inner coherency of the position adopted was a posteriori analyzed. We demonstrated that, at the third order, the corrections due to the pressure gradients and streamwise viscous diffusion, foreseen by the steady Navier-Stokes model, are still negligible: the pressure contribution identically vanishes and the longitudinal diffusion still remains from two to three order of magnitude lower than the lateral diffusion and the convective transport.

We analyzed the asymptotic behaviour of our expansion in the lateral far field, demonstrating that at finite values of x the coefficient function $\phi(\eta)$ for the streamwise velocity decays to zero as a Gaussian law for $n = 1$ and as a power law of exponent -2 for $n = 2$ and of exponent -3 for $n \geq 3$, while the coefficient function $\chi(\eta)$ for the cross-stream velocity goes to zero for $n = 0, 1$ and to a constant value for $n \geq 2$, that leaves v to vanish as $x^{-3/2}$ for $x \to \infty$. When $x \to \infty$ our solution coincides with the Gaussian representation given by the Oseen approximation. According to these solutions the asymptotic form of Prandtl's equations with zero longitudinal pressure gradient confirms the physical expectation of a lateral diffusion becoming progressively smaller in comparison with the non linear convection. This expansion foresees the entrainment of ambient fluid from the sides and agrees with the interpretation of the field at large distances from the body as a superposition of a uniform flow and of a source-like flow placed at the center of the body to compensate for the mass flux deficit in the wake. The middle and far wake regions so described are in excellent contrast with the experimental data available in the middle wake of the circular cylinder by KOVÁSZNAY [14]. We consider these expansion solutions very useful to represent the diverging basic flow of the middle and far wake in application concerning the non parallel stability analysis.

Appendix

A Coefficients of the transformed equation (17)

A.1 The functions $P_n(\eta)$ and $S_n(\eta)$

Function P_n is the second coefficient of the transformed equation (17), and is obtained directly from the substitution (16) in eq. (15). It is an even function that has $n-1$ simple poles. The general formula can be expressed by means of the Hermite polynomials $\mathrm{Hr}_n(\eta) = \mathrm{H}_n[\frac{1}{2}R^{1/2}\eta]$:

$$P_{2m}(\eta) = -R^{1/2}\ \frac{\mathrm{Hr}_{2m}(\eta) - \frac{1}{2}\ R^{1/2}\eta\ \mathrm{Hr}_{2m-1}(\eta)}{\mathrm{Hr}_{2m-1}(\eta)}\ , \tag{72}$$

$$P_{2m+1}(\eta) = -R^{1/2}\ \frac{\mathrm{Hr}_{2m+1}(\eta) - 4n\ \mathrm{Hr}_{2m-1}(\eta)}{\mathrm{Hr}_{2m}(\eta)}\,. \tag{73}$$

The asymptotic behaviour is $P_n \sim -(R/2)\,\eta \to \mp\infty$ as $\eta \to \pm\infty$. Function P_n is rational (a polynomials ratio), so it is always analytically integrable, what leads to the expressions

$$S_{2m}(\eta) = \int P_{2m}(\eta)\ d\eta = \ln\big[R^{-1}\mathrm{Hr}^2_{2m-1}(\eta)\big] - \tfrac{1}{4}\ R\eta^2\,, \tag{74}$$

$$S_{2m+1}(\eta) = \int P_{2m+1}(\eta)\ d\eta = \ln\big[\mathrm{Hr}^2_{2m}(\eta)\big] - \tfrac{1}{4}\ R\eta^2\,, \tag{75}$$

$$\mathrm{e}^{\pm S_{2m}(\eta)} = R^{\mp 1}\mathrm{Hr}^{\pm 2}_{2m-1}(\eta)\ \mathrm{e}^{\mp(R/4)\,\eta^2}\,, \tag{76}$$

$$\mathrm{e}^{\pm S_{2m+1}(\eta)} = \mathrm{Hr}^{\pm 2}_{2m}(\eta)\ \mathrm{e}^{\mp(R/4)\,\eta^2}\,. \tag{77}$$

A.2 The functions $T_n(\eta)$ and $Q_n(\eta)$

Function T_n is the inhomogeneous term in eq. (15). It is obtained directly by using the expressions (9) in the boundary-layer equation (1), it is smooth like u and v.

A simplification in espression (13) is made possible noting that $-(\eta/2)\,\phi'_{n-1}\phi_1$ and $\phi'_{n-1}\chi_1$ are equal and opposite. Thus, T_n may be rewritten as

$$T_n = -\frac{n}{4}\sum_{i=1}^{n-1}\phi_i\phi_{n-i} + \sum_{i=1}^{n-2}\Big(-\frac{\eta}{2}\ \phi'_i\phi_{n-i} + \phi'_i\chi_{n-i}\Big)\ . \tag{78}$$

A useful assumption is to factor the constant A_n from the T_n function, writing

$$T_n = A^n\tilde{T}_n\,. \tag{79}$$

Constant A is defined as a proportionality coefficient for ϕ_1, i.e. $\phi_1 \propto A$, and also $\chi_1 \propto A$. Thus, we have $T_2 \propto \phi_1^2 \propto A^2$, and this gives $\phi_2 \propto A^2$ and $\chi_2 \propto A^2$. Then, we obtain $T_3 \propto A^3$, and so on; the property $T_n \propto A^n$ can be stated by the induction method. From relation (78) it is possible to show that $T_n \propto A^n\ \mathrm{e}^{-(R/2)\,\eta^2}$. In spite to this proportionality relation, in B it will be shown that the asymptotic behaviour of T_n is in general of the kind $T_n \sim \eta^{-3}$ as $|\eta| \to \infty$ for $n \geq 3$, whilst $T_2 \sim \mathrm{e}^{-(R/2)\,\eta^2}$. T_n is in general an odd function, as can be deduced from the original partial differential equations.

The function Q_n is defined by the relation (18), so

$$Q_n(\eta) = R\ \frac{T_n(\eta)}{\partial^{n-1}_\eta\gamma(\eta)} \propto \frac{T_n(\eta)}{\mathrm{Hr}_{n-1}(\eta)}\ , \tag{80}$$

and this shows that Q_n has $n-1$ simple poles.

B Algebraic decay of the solutions

Function ϕ_2, calculated in § 3.1, when expanded in separate terms becomes

$$\phi_2 = A^2C_2\ \mathrm{e}^{-(R/4)\,\eta^2}\ {}_1\mathrm{F}_1(-\tfrac{1}{2},\tfrac{1}{2};(R/4)\,\eta^2) - \tfrac{1}{2}\,A^2\ \mathrm{e}^{-(R/2)\,\eta^2} - \tfrac{1}{4}\,A^2(\pi R)^{1/2}\ \mathrm{e}^{-(R/4)\,\eta^2}\ \eta\ \ \mathrm{erf}\big(\tfrac{1}{2}R^{1/2}\eta\big)\,; \tag{81}$$

The asymptotic behaviour of this function as $|\eta| \to \infty$ is determined by the term

$$A^2C_2\ \mathrm{e}^{-(R/4)\,\eta^2}\ {}_1\mathrm{F}_1(-\tfrac{1}{2},\tfrac{1}{2};(R/4)\,\eta^2) \sim [A^2C_2\ \mathrm{e}^{-(R/4)\,\eta^2}]\ [-\tfrac{1}{2}\,\mathrm{e}^{(R/4)\,\eta^2}\ \big((R/4)\,\eta^2\big)^{-1}] \sim \eta^{-2}\,; \tag{82}$$

and the behaviour of the whole function may written as

$$\phi_2 \sim \mathcal{C}C_2\eta^{-2} + \mathcal{C}\eta E + \mathcal{C}E^2\,, \tag{83}$$

where we introduce the short notation $E = \mathrm{e}^{-(R/4)\,\eta^2}$ and the symbol $\mathcal{C}$ to represent a generic asymptotic constant. We see that ϕ_2 could satisfy a 'constraint of rapid decay' only in the particular case of a constant $C_2 = 0$, as done in the paper by GOLDSTEIN [10], and this would lead to $\phi_2 \sim nE$; otherwise, the behaviour is $\phi_2 \sim \eta^{-2}$. The behaviour of the function χ_2 is given by formula (10):

$$\chi_2 = \tfrac{1}{2}\,\eta\phi_2 + \tfrac{1}{2}\int \phi_2\, d\eta\,. \tag{84}$$

When ϕ_2 is integrated, the secondary term $\mathcal{C}E^2$, which is a Gaussian function, generates an error function term. Thus χ_2 behaves like a constant as $|\eta| \to \infty$, independently of the value of the constant C_2, i.e., χ_2 goes as a constant in both cases of ϕ_2 Gaussian or algebraic. This shows also that, in general, in a correct study of the asymptotic properties of ϕ_n we must retain not only the dominant term, that is usually algebraic, but also the terms that, after an integration, produce asymptotic constants. The behaviour of χ_2 is a general property, since it is shown by every χ_n with $n > 2$, as we will see. This property is related to the phenomenon of flow entrainment toward the center of the wake.

After the second order, it is useful to develop a general method to study the behaviour of the generic ϕ_n, because this functions depend on the integrals F_n, that cannot be determined explicitly when $n > 2$: first, we write ϕ_n in the form

$$\phi_n(\eta) = A^n C_n\, \mathrm{e}^{-(R/4)\,\eta^2}\, {}_1\mathrm{F}_1\left(\frac{1-n}{2}, \tfrac{1}{2}; (R/4)\,\eta^2\right) + RA^n\, \mathrm{e}^{-(R/4)\,\eta^2}\, \mathrm{Hr}_{n-1}(\eta)\, F_n(\eta) = \phi_{An} + \phi_{Bn}\,, \tag{85}$$

so that the function is splitted in two parts that can be considered separately. The first part ϕ_{An} contains as a factor the confluent hypergeometric function and it shows different asymptotic behaviours depending on the parity of n:

$$\phi_{An} \sim E\, {}_1\mathrm{F}_1\left(\frac{1-n}{2}, \frac{1}{2}; (R/4)\,\eta^2\right) \sim \begin{cases} E\,\eta^{-n}E^{-1} \sim \eta^{-2m} & \text{for} \quad n = 2m \\ \eta^{2m}E & \text{for} \quad n = 2m+1 \end{cases}, \tag{86}$$

the latter being due to a property of the hypergeometric function (if a is an integer, ${}_1\mathrm{F}_1(-a, b, z)$ becomes a polynomial of order a). This is a first cause for the algebraic decay as $|\eta| \to \infty$ of ϕ_n, that could be eliminated simply by setting $C_{2m} = 0$. But we will see that also ϕ_{Bn} has an algebraic decay, and this part of ϕ_n cannot be manipulated because does not carry as factor integration constants. Its asymptotic behaviour is of the kind

$$\mathrm{e}^{-(R/4)\,\eta^2}\, \mathrm{Hr}_{n-1}(\eta)\, F_n(\eta) \sim \eta^{n-1} F_n E\,. \tag{87}$$

Integral F_n involves G_n, as shown by eq. (27):

$$F_n = \int E^{-1}\mathrm{Hr}_{n-1}^{-2} G_n\, d\eta \sim \int \eta^{-2n+2} E^{-1} G_n\, d\eta\,. \tag{88}$$

Let us start from the study of G_n, written explicitly using formula (13):

$$\begin{aligned} A^n G_n &= \int \mathrm{Hr}_{n-1} T_n\, d\eta \\ &\sim \int \eta^{n-1}\left[-\frac{n}{4}\sum_{i=1}^{n-1} \phi_i\phi_{n-i} + \sum_{i=1}^{n-2}\left(-\frac{\eta}{2}\,\phi_i'\phi_{n-i} + \phi_i'\chi_{n-i}\right)\right] d\eta\,. \end{aligned} \tag{89}$$

Suppose to know the asymptotic behaviours of the previous orders functions, that are in general represented by sums of the kind (83). By substituting for these expressions in the products $\phi_i\phi_{n-i}, \phi_i'\phi_{n-i}, \phi_i'\chi_{n-i}$, we will find a new sum, involving several combination of the previous terms with their derivatives. The result must be multiplied by the polynomial $\mathrm{Hr}_{n-1} \sim \eta^{n-1}$, and integrated once. This will lead to another sum of terms, that represents the asymptotic behaviour of G_n, and that can be used to determine the behaviour of F_n, defined by eq. (88). After the second integration we can use eq. (87) to obtain the final expression that represents the asymptotic behaviour of ϕ_{Bn}. Finally, the behaviour of ϕ_n will be determined by adding the term coming from ϕ_{An}, given by (86). It is worth noting that here the integrations are to be performed asymptotically; the required integration rules are shown in Table 1 (the determination of integrals involving Gaussian functions is not immediate. See for example ABRAMOWITZ and STEGUN [1], chapters 5 and 7).

This method is clearly explained by the example corresponding to $n = 3$. The behaviours of ϕ_1, ϕ_2, χ_1, χ_2 bring into T_3 several combination of the terms listed above:

$$\begin{aligned} &\phi_1 \sim E\,, \\ &\phi_2 \sim \mathcal{C}C_2\eta^{-2} + \mathcal{C}\eta E + \mathcal{C}E^2\,, \\ &\chi_1 \sim \eta E\,, \\ &\chi_2 \sim \mathcal{C}\,, \end{aligned} \tag{90}$$

Table 1: Asymptotic integration rules

$$\int \eta^k\, d\eta \sim \begin{cases} \eta^{k+1} & \text{for} \quad k \neq -1 \\ \ln|\eta| & \text{for} \quad k = -1 \end{cases},$$

$$\int \eta^k E^j\, d\eta \sim \begin{cases} c & \text{for} \quad k \text{ even},\ j > 0 \\ \eta^{k-1}E^j & \text{for} \quad k \text{ odd},\ j > 0 \end{cases},$$

$$\int \eta^k E^{-j}\, d\eta \sim \eta^{k-1}E^{-j}\ (j > 0)\,.$$

Substitute for these asymptotic behaviours in (89), keeping all the terms, and multiplying by Hr_2. By integrating once, we find

$$G_3 \sim (\mathcal{C}C_2 + \mathcal{C}) + \mathcal{C}C_2\eta E + (\text{other terms} \sim \eta^k E^j,\ j > 0)\,, \tag{91}$$

that represents the asymptotic behaviour of G_3. This goes into F_3, according to formula (88). After a second integration we have

$$F_3 \sim (\mathcal{C}C_2 + \mathcal{C})\,\eta^{-5}E^{-1} + \mathcal{C}C_2\eta^{-2} + (\text{other terms} \sim \eta^k E^j, j \geq -1)\,. \tag{92}$$

Substituting for F_3 in the eq. (87) we get the asymptotic behaviour of ϕ_{B3}:

$$\phi_{B3} \sim (\mathcal{C}C_2 + \mathcal{C})\,\eta^{-3} + \mathcal{C}C_2 E + (\text{terms} \sim \eta^{-5}) + (\text{other terms} \sim \eta^k E^j, j > 0)\,. \tag{93}$$

From relation (86) we know that $\phi_{A3} \sim \eta^2 E$, so we may express the asymptotic behaviour of ϕ_3 as

$$\phi_3 \sim (\mathcal{C}C_2 + \mathcal{C})\,\eta^{-3} + \mathcal{C}C_2 E + \mathcal{C}\eta^2 E + (\text{terms} \sim \eta^{-5}) + (\text{other terms} \sim \eta^k E^j, j > 0)\,. \tag{94}$$

Note that, for brevity reasons, in the asymptotic formulae to keep the parity of the solutions, the sign of the constant factors of odd powers of η, here generically called $\mathcal{C}$, may change switching from $\eta \to \infty$ to $\eta \to -\infty$.
The above result shows that $\phi_3 \sim \eta^{-3}$ for every value of C_2, i.e. whether or not ϕ_2 has a Gaussian decay. More, as an effect of the integration of the terms $\sim E, \eta^2 E$, we see from Table 1 that $\chi_3 = (\eta/2)\,\phi_3 + \int \phi_3\, d\eta$ behaves as a constant.

For the following orders, supposing $C_2 \neq 0$, it is no more necessary to retain explicitly the constant C_2 in the calculations, because now we know that it is impossible to obtain a complete series with Gaussian behaviour. Following the same method, the results obtained for the orders 4 and 5 are

$$\phi_4 \sim \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{-2}E + (\text{terms} \sim \eta^{-k},\ k > 3) + (\text{terms} \sim \eta^{-k}\ln|\eta|,\ k > 3) + (\text{terms} \sim \eta^k E^j, j > 0)\,, \tag{95}$$

$$\chi_4 \sim \mathcal{C} + (\text{secondary terms})\,, \tag{96}$$

$$\phi_5 \sim \mathcal{C}\eta^{-3} + \mathcal{C}\eta^4 E + (\text{terms} \sim \eta^{-k},\ k > 3) + (\text{terms} \sim \eta^{-k}\ln|\eta|,\ k > 4) + (\text{terms} \sim \eta^k E^j, j > 0)\,, \tag{97}$$

$$\chi_5 \sim \mathcal{C} + (\text{secondary terms})\,. \tag{98}$$

This leads to a simple hypothesis for the generic order $i \geq 3$:

$$\phi_i \sim \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{2m}E + (\text{other secondary terms})\,, \tag{99}$$

$$\chi_i \sim \mathcal{C} + (\text{secondary terms})\,, \tag{100}$$

The term $\sim \eta^{2m}E$ in (99) leads to (100), because of the integration involved in the latter equation, see Table 1. This explains why it is necessary to evidence at least one term showing this property.

The above behaviours can be stated by the induction method. It is necessary to suppose that relations (99) and (100) hold for $3 \leq i < n$. By using the behaviours of $\phi_1, \phi_2, \chi_1, \chi_2$ (eqs. (90)) as a starting condition, it can be proved that properties (99), (100) hold also for the order n.

First, study the behaviour of the generic T_n, as given by (13). This formula may be splitted in three parts, namely

$$T_{an} \sim \sum_{i=1}^{n-1} \phi_i \phi_{n-i}\,, \qquad T_{bn} \sim \sum_{i=1}^{n-2} \eta \phi_i' \phi_{n-i}\,, \qquad T_{cn} \sim \sum_{i=1}^{n-2} \phi_i' \chi_{n-i}\,. \tag{101}$$

Their behaviour may be determined using the hypotheses and the starting conditions. By retaining only the dominant terms we have:

$$\begin{aligned} T_{an} &\sim \phi_1\phi_{n-1} + \phi_2\phi_{n-2} + \ldots + \phi_i\phi_{n-i} + \ldots + \phi_2\phi_{n-2} + \phi_1\phi_{n-1} \\ &\sim \mathcal{C}\eta^{-3}E + \mathcal{C}\eta^{-5} + \ldots + c_i\eta^{-6} + \ldots \sim \eta^{-5} \end{aligned} \tag{102}$$

(note that this sum is symmetric, so that only half of the terms is to be considered), and, in a similar way,

$$T_{bn} \sim \eta^{-5}\,, \qquad T_{cn} \sim \eta^{-3}\,. \tag{103}$$

Thus, $T_n \sim T_{cn} \sim \eta^{-3}$, so that, substituting (89) in we find $G_n \sim \eta^{n-3}$, and using then (88), we obtain $F_n \sim \eta^{-n-2}E^{-1}$. Finally, (87) gives

$$\phi_{Bn} \sim \eta^{-3}\,. \tag{104}$$

Now, we know also the behaviour of ϕ_{An} from relation (86), so we may write

$$\phi_n \sim \begin{cases} \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{n-1}E + \ldots & \text{for } n \text{ odd} \\ \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{-n} + \ldots & \text{for } n \text{ even} \end{cases} \tag{105}$$

(here and in the followings the dots stand for other secondary terms with respect to η^{-3}), and the thesis is proved only for n odd. In order to extend the result to even n, we go back to the study of T_n, searching for secondary terms that may give rise almost to one term of the kind $\eta^{2m}E$ in ϕ_n when n is even. To do this, we note that the behaviour of the second term in T_{an} (see (102)), if expanded, is

$$\phi_2\phi_{n-2} \sim \mathcal{C}\eta^{-5} + \mathcal{C}\eta^{-2}E + \ldots\,, \tag{106}$$

note that the new term is not necessarily the second in order of magnitude. We are instead interested to evidence the presence of a term of this particular kind, necessary to ensure that $\chi \sim c$. Relation (106) leads to

$$G_n \sim \mathcal{C}\eta^{n-3} + \mathcal{C}\eta^{n-4}E + \ldots\,, \tag{107}$$

$$F_n \sim \mathcal{C}\eta^{-n-2}E^{-1} + \mathcal{C}\eta^{-n-1} + \ldots\,, \tag{108}$$

$$\phi_{Bn} \sim \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{-2}E + \ldots\,, \tag{109}$$

and finally to

$$\phi_n \sim \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{-2}E + \ldots \quad \text{for } n \text{ even.} \tag{110}$$

This completes the proof, since we may write now that for $C_2 \neq 0$ and $n \geq 3$

$$\phi_n \quad \sim \quad \begin{cases} \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{n-1}E + \ldots & \text{for } n \text{ odd} \\ \mathcal{C}\eta^{-3} + \mathcal{C}\eta^{-2}E + \ldots & \text{for } n \text{ even} \end{cases}, \tag{111}$$

$$\chi_n \sim \mathcal{C} + (\text{other secondary terms}), \tag{112}$$

the latter relation being given by the former through integration. This imply a global lateral asymptotic behaviour of the asymptotic expansion of the kind $\phi_n \sim \mathcal{C}\eta^{-2}$ and $\chi_n \sim \mathcal{C}$ for any value of $C_2 \neq 0$. Note that the second term of ϕ_n may be rewritten in the form $\eta^{2m}E$, with $m = (n-1)/2m$ for n odd and $m = -1$ for n even.

The case of $C_2 = 0$ – a particular choice, since it is almost impossible that the experimental boundary condition (6) may lead to such a value – induces the disappearing of many secondary terms from the asymptotic relationships above considered. However, since $\phi_3 \sim \eta^{-3}$ independently of the value of C_2, it is possible to demonstrate that in this case, when $n \geq 4$, relations (110) and (111) become

$$\phi_n \sim \mathcal{C}\eta^{-4} + \mathcal{C}\eta^{2m}E + \ldots, \tag{113}$$

$$\chi_n \sim \mathcal{C} + \ldots, \tag{114}$$

the latter relation being again given by the former through integration. This imply in its turn a global lateral decay of the expansion for $C_2 = 0$ of the kind $\phi_n \sim \mathcal{C}\eta^{-3}$ and $\chi_n \sim \mathcal{C}$.

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