Initial-value problem for shear flows

1 Mathematical Framework

The early transient and long asymptotic behaviour is studied using the initial-value problem formulation for two typical shear flows, the plane Poiseuille flow and the bluff-body wake (see Fig. 1b and 1c, respectively). The continuity and Navier-Stokes equations that describe the perturbed system, subject to small three-dimensional disturbances, are linearized and written as

$$\frac{\partial \widetilde{u}}{\partial x} + \frac{\partial \widetilde{v}}{\partial y} + \frac{\partial \widetilde{w}}{\partial z} = 0, \tag{1}$$

$$\frac{\partial \widetilde{u}}{\partial t} + U\frac{\partial \widetilde{u}}{\partial x} + \widetilde{v}\frac{\partial U}{\partial y} + \frac{\partial \widetilde{p}}{\partial x} = \frac{1}{Re}\nabla^2 \widetilde{u}, \tag{2}$$

$$\frac{\partial \widetilde{v}}{\partial t} + U\frac{\partial \widetilde{v}}{\partial x} + \frac{\partial \widetilde{p}}{\partial y} = \frac{1}{Re}\nabla^2 \widetilde{v}, \tag{3}$$

$$\frac{\partial \widetilde{w}}{\partial t} + U\frac{\partial \widetilde{w}}{\partial x} + \frac{\partial \widetilde{p}}{\partial z} = \frac{1}{Re}\nabla^2 \widetilde{w}, \tag{4}$$

where $(\widetilde{u}(x,y,z,t), \widetilde{v}(x,y,z,t), \widetilde{w}(x,y,z,t))$ and $\widetilde{p}(x,y,z,t)$ are the perturbation velocity components and pressure, respectively. U and dU/dy indicate the base flow profile (under the near-parallelism assumption) and its first derivative in the shear direction. For the channel flow, the independent spatial variable z is defined from $-\infty$ to $+\infty$, the x variable from $-\infty$ to $+\infty$, and the y from -1 to 1. For the wake flow, z is defined from $-\infty$ to $+\infty$, x from 0 to $+\infty$, and y from $-\infty$ to $+\infty$. All the physical quantities are normalized with respect to a typical velocity (the free stream velocity, U_f, and the centerline velocity, U_0, for the 2D wake and the plane Poiseuille flow, respectively), a characteristic length scale (the body diameter, D, and the channel half-width, h, for the 2D wake and the plane Poiseuille flow, respectively), and the density, ρ.

The base flow of the wake is approximated at an intermediate ($x_0 = 10$) and at a far longitudinal station ($x_0 = 50$), through two-dimensional analytical expansion

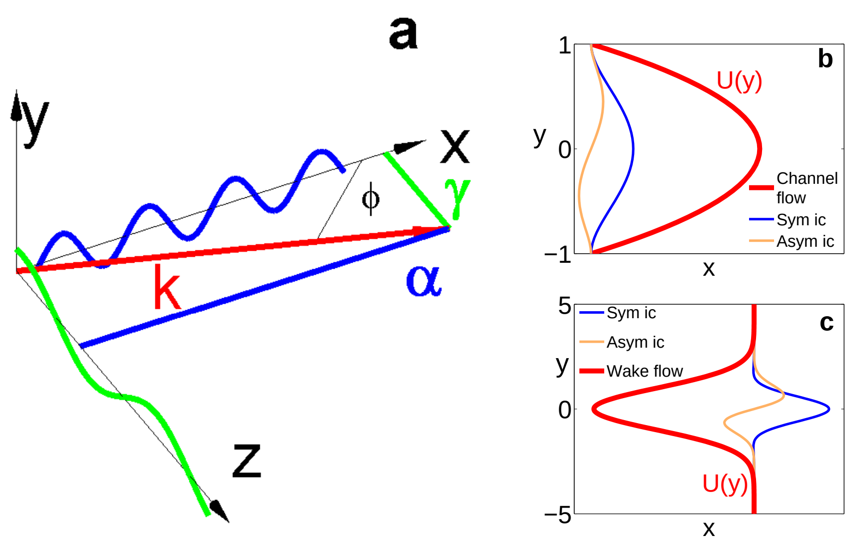

Figure 1: (a) Perturbation geometry scheme. The perturbation propagates in the direction of the polar wavenumber, $k = \sqrt{\alpha^2 + \gamma^2}$. ϕ is the angle of obliquity with respect to the basic flow. (b)-(c) Symmetric and antisymmetric initial conditions in terms of the perturbation transversal velocity, $\hat{v}(y, t = 0)$ (thin curves), and base flow velocity profiles, $U(y)$ (thick curves).

solutions [1] of the Navier-Stokes equations. Assuming that the bluff-body wake is a slowly evolving spatial system, the base flow is frozen at each longitudinal station past the body, by using the first orders of the expansion solutions [1],

$$U(y; x_0, Re) = 1 - a x_0^{-1/2} \mathrm{e}^{-\frac{\mathrm{Re}}{4}\frac{\mathrm{y}}{\mathrm{x_0}}},$$

where a is related to the drag coefficient C_D ($a = \frac{1}{4}(Re/\pi)^{1/2} c_D(Re)$, see [1]), and x_0 is the streamwise longitudinal station.

The plane channel flow is homogeneous in the x direction and is represented by the Poiseuille solution, $U(y) = 1 - y^2$.

By combining equations (1) and (4) to eliminate the pressure terms, the linearized equations describing the perturbation dynamics become

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2\widetilde{v}-\frac{\partial\widetilde{v}}{\partial x}\frac{d^2U}{dy^2} = \frac{1}{Re}\nabla^4\widetilde{v}, \tag{5}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\widetilde{\omega}_y+\frac{\partial\widetilde{v}}{\partial z}\frac{dU}{dy} = \frac{1}{Re}\nabla^2\widetilde{\omega}_y, \tag{6}$$

where $\widetilde{\omega}_y$ is the transversal component of the perturbation vorticity. The physical quantity $\widetilde{\Gamma}$ is defined as

$$\nabla^2\widetilde{v}=\widetilde{\Gamma}. \tag{7}$$

In so doing, the three coupled equations (5), (6) and (7) describe the perturbed system. Equations (5) and (6) are the Orr-Sommerfeld and Squire equations, respectively, which are obtained from the classical linear stability analysis, here they are written for three-dimensional disturbances in partial differential equation form. On the basis of kinematics, one can derive the relation

$$\widetilde{\Gamma}=\frac{\partial\widetilde{\omega}_z}{\partial x}-\frac{\partial\widetilde{\omega}_x}{\partial z}, \tag{8}$$

that physically links the perturbation vorticity in the x and z directions ($\widetilde{\omega}_x$ and $\widetilde{\omega}_z$, respectively) to the perturbation velocity field through Eq. (7). If equations (5) and (7) are combined, one gets the following equation

$$\frac{\partial\widetilde{\Gamma}}{\partial t}+U\frac{\partial\widetilde{\Gamma}}{\partial x}-\frac{\partial\widetilde{v}}{\partial x}\frac{d^2U}{dy^2}=\frac{1}{Re}\nabla^2\widetilde{\Gamma}, \tag{9}$$

which, together with (6) and (7), fully describe the perturbed system in terms of vorticity and velocity [2, 3, 4, 5].

The perturbations are Fourier transformed in the x and z directions for the channel flow. Two real wavenumbers, α and γ, are introduced along the x and z coordinates, respectively. A combined Laplace-Fourier decomposition is performed for the wake flow in the x and z directions. In this case, a complex wavenumber $\alpha=\alpha_r+i\alpha_i$ can be introduced along the x coordinate, as well as a real wavenumber, γ, along the z coordinate. The perturbation quantities $(\widetilde{v},\widetilde{\Gamma},\widetilde{\omega}_y)$ involved in the system dynamics are now indicated as $(\hat{v},\hat{\Gamma},\hat{\omega}_y)$, where

$$\hat{f}(y,t;\alpha,\gamma)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{f}(x,y,z,t)e^{-i\alpha x-i\gamma z}dxdz, \tag{10}$$

indicates in the $\alpha - \gamma$ phase space the two-dimensional Fourier transform (in the case of the channel flow) of a general dependent variable, $\widetilde{f}$, and

$$\hat{g}(y,t;\alpha,\gamma) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \widetilde{g}(x,y,z,t)e^{-i\alpha x - i\gamma z}dxdz, \tag{11}$$

indicates the two-dimensional Laplace-Fourier transform (in the case of the wake flow) of a general dependent variable, $\widetilde{g}$. To obtain a finite perturbation kinetic energy, the imaginary part, α_i, of the Laplace transformed complex longitudinal wavenumber can only assume non-negative values and can thus be defined as a spatial damping rate in the streamwise direction. In so doing, perturbative waves can spatially decay ($\alpha_i > 0$) or remain constant in amplitude ($\alpha_i = 0$). Here, for the sake of simplicity, we have $\alpha_i = 0$, therefore $\alpha = \alpha_r$. The governing partial differential equations we consider are thus

$$\frac{\partial^2 \hat{v}}{\partial y^2} - k^2\hat{v} = \hat{\Gamma}, \tag{12}$$

$$\frac{\partial \hat{\Gamma}}{\partial t} = -ikcos(\phi)U\hat{\Gamma} + ikcos(\phi)\frac{d^2U}{dy^2}\hat{v} + \frac{1}{Re}\left(\frac{\partial^2\hat{\Gamma}}{\partial y^2} - k^2\hat{\Gamma}\right), \tag{13}$$

$$\frac{\partial \hat{\omega}_y}{\partial t} = -ikcos(\phi)U\hat{\omega}_y - iksin(\phi)\frac{dU}{dy}\hat{v} + \frac{1}{Re}\left(\frac{\partial^2\hat{\omega}_y}{\partial y^2} - k^2\hat{\omega}_y\right), \tag{14}$$

where $\phi = tan^{-1}(\gamma/\alpha)$ is the perturbation obliquity angle with respect to the x-y plane, $k = \sqrt{\alpha^2+\gamma^2}$ is the polar wavenumber and $\alpha = kcos(\phi)$, $\gamma = ksin(\phi)$ are the wavenumber components in the x and z directions, respectively. See Fig. 1a, where the perturbation scheme is reported.

Various initial conditions can be used to explore the transient behavior. The important feature here is the ability to make arbitrary specifications. It is physically reasonable to assume that the natural issues affecting the initial conditions are the symmetry and the spatial lateral distribution of disturbances. It has been observed [6, 7, 8] that, keeping all the other parameters fixed, if the perturbation oscillates rapidly or mainly lies outside the shear region then, for a stable configuration, the final damping is accelerated while, for an unstable configuration, the asymptotic growth is delayed. However, the general qualitative scenario is not altered. Therefore, to perform a more synthetic perturbative analysis, we only focus on symmetric and asymmetric inputs which are localized and distributed over the whole shear region (see Fig. 1b-c). The transversal vorticity $\hat{\omega}_y(y,t)$

	Channel flow	Wake flow
$\hat{v}(y, t=0)$	$\Omega(\alpha,\gamma)(1-y^2)^2$ or $\Omega(\alpha,\gamma)y(1-y^2)^2$	$\Omega(\alpha,\gamma)exp(-y^2)cos(y)$ or $\Omega(\alpha,\gamma)exp(-y^2)sin(y)$
$\hat{\omega}_y(y, t=0)$	0	0

Table 1: S1. Initial conditions for the channel and wake flows.

is initially taken equal to zero to highlight the three-dimensionality net contribution on its temporal evolution. The effects of non-zero initial conditions on the transversal vorticity $\hat{\omega}_y(y,t)$ can be found in [7, 8]. The imposed initial conditions are reported in table 1, for the channel and wake flows. $\Omega(\alpha,\gamma)$ is the phase space transform of the x-z dependence prescribed at time $t = 0$. Here, we set $\Omega(\alpha,\gamma) = 1$, which means that no wavenumber is initially biased in the phase space.

For the channel flow no-slip and impermeability boundary conditions are imposed,

$$\hat{v}(y=\pm 1,t) = \frac{\partial \hat{v}}{\partial y}(y=\pm 1,t) = \hat{\omega}_y(y=\pm 1,t) = 0, \tag{15}$$

while for the wake flow uniformity at infinity and finiteness of the energy are imposed,

$$\hat{v}(y\to\pm\infty,t) = \frac{\partial \hat{v}}{\partial y}(y\to\pm\infty,t) = \hat{\omega}_y(y\to\pm\infty,t) = 0. \tag{16}$$

In order to measure the growth of the perturbations, we define the kinetic energy density, e,

$$\begin{aligned} e(t;\alpha,\gamma) &= \frac{1}{2}\int_{-y_f}^{+y_f}(|\hat{u}|^2+|\hat{v}|^2+|\hat{w}|^2)dy \\ &= \frac{1}{2}\frac{1}{(\alpha^2+\gamma^2)}\int_{-y_f}^{+y_f}\left(\left|\frac{\partial \hat{v}}{\partial y}\right|^2+(\alpha^2+\gamma^2)|\hat{v}|^2+|\hat{\omega}_y|^2\right)dy, \end{aligned} \tag{17}$$

where $-y_f$ and y_f are the computational limits of the domain, while $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{\omega}_y$ are the transformed velocity and transversal vorticity components of the perturbation field, respectively. We can also define the amplification factor, G, as the kinetic energy density normalized with respect to its initial value,

$$G(t;\alpha,\gamma) = \frac{e(t;\alpha,\gamma)}{e(t=0;\alpha,\gamma)}. \tag{18}$$

Assuming that the temporal asymptotic behavior of the linear perturbations is exponential, the temporal growth rate, r, can be defined as

$$r(t;\alpha,\gamma) = \frac{log[e(t;\alpha,\gamma)]}{2t}, \qquad t>0. \tag{19}$$

This quantity has a precise meaning when the asymptotic state is reached, that is, when it becomes a constant.

The frequency, ω, of the perturbation is defined as the temporal derivative of the unwrapped wave phase, $\theta(y,t;\alpha,\gamma)$, at a specific spatial point along the y direction. The wrapped phase,

$$\theta_w(y,t;\alpha,\gamma) = arg(\hat{v}(y,t;\alpha,\gamma)), \tag{20}$$

is a discontinuous function of t with $\theta_w \in (-\pi,+\pi]$, while the unwrapped phase, θ, is a continuous function defined by changing absolute θ_w jumps greater than or equal to π to their 2π complement. In the case of the wake we use as reference observation point $y = y_0 = 1$, and in the case of the channel flow the point $y = y_0 = 0.5$. The frequency [6] is thus

$$\omega(t;y_0,\alpha,\gamma) = |d\theta(t;y_0,\alpha,\gamma)|/dt. \tag{21}$$

2 Numerical method, flow chart and Matlab scripts

Equations (12)-(14) are numerically solved by the method of lines [9]: the equations are first discretized in the spatial domain and then integrated in time. The spatial derivatives in the y domain are discretized using a second-order finite difference scheme for the first and second derivatives. One-sided differences are adopted at the boundaries, while central differenced derivatives are used in the remaining part of the domain. The spatial grid is uniform with a spatial step, h, which is equal to 0.05 and 0.004, for the wake and channel flows, respectively. Since the wake flow is spatially unbounded in the transversal direction, the spatial domain, $[-y_f, y_f]$, is chosen so that the numerical solutions are insensitive to further extensions of the computational domain size. The spatial domain is enlarged when long perturbative waves are analyzed, thus we put

- Channel flow: $y \in [-1, 1]$, $h = 0.004$, grid points $N = 501$;
- Wake flow and $k > 1$: $y \in [-20, 20]$, $h = 0.05$, grid points $N = 801$;
- Wake flow and $k \in [0.75, 1]$: $y \in [-30, 30]$, $h = 0.05$, grid points $N = 1201$;
- Wake flow and $k \in [0.45, 0.7]$: $y \in [-40, 40]$, $h = 0.05$, grid points $N = 1601$.

Equations (12)-(14) are then integrated in time by means of an adaptative one-step solver, based on an explicit Runge-Kutta (2,3) formula and implemented in the `ode23` Matlab function [10, 11]. The choice of the `ode23` routine is a good compromise between nonstiff solvers (`ode45` and `ode113`), which give a higher order of accuracy, and stiff solvers (`ode15s` and `ode23s`), which can in general be more efficient.

In the remaining part of this Section, we describe the structure of the numerical code (see Fig. 2, where the flowchart is shown) and the details of the Matlab scripts:

- `channel_main.m/wake_main.m`: this file contains the main program necessary to use the code. The simulation parameters (angle of obliquity, symmetry of the perturbations, range of wavenumbers) are set as well as base flow configurations (Reynolds numbers and longitudinal downstream station for the wake flow) are set. A loop is defined which begins with the first simulated wavenumber, k_{in}, and ends with the last simulated wavenumber, k_{fin}. Once this cycle is entered, for a fixed wavenumber, the function `IVP_complete.m` is called;
- `IVP_complete.m`: the perturbative equations (12)-(14) are solved in this routine. Once the initial conditions are defined, the `ode23` function is iteratively called together with three auxiliary functions, `dhdt.m`, `MM1.m` and `solve_for_v_complete.m`. The `dhdt.m` script represents the right-hand side of the perturbative equations (13)-(14), while `MM1.m` and `solve_for_v_complete.m` link the solutions of Eq. (12) to (13) (`MM1.m` is called for the channel flow, `solve_for_v_complete.m` for the wake

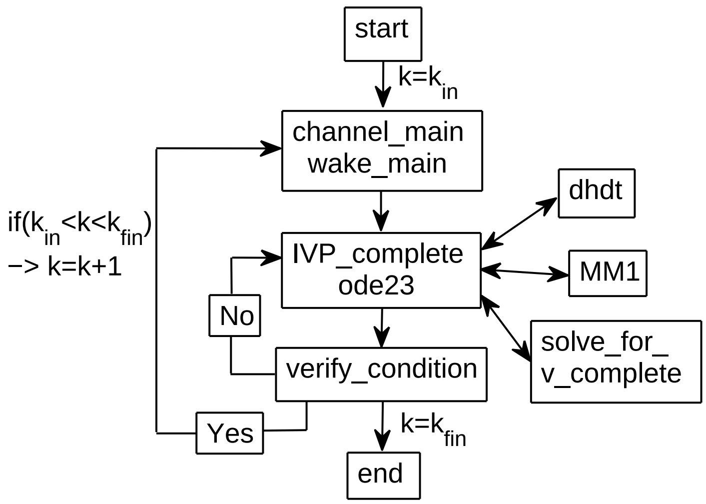

Figure 2: Flowchart of the Matlab codes.

flow). Once the solutions are obtained, the perturbation velocity field and energy are computed, and `verify_condition.m` is called;

- `verify_condition.m`: this function verifies if the asymptotic condition for the current perturbative wave is reached. If the asymptotic state is not reached, `IVP_complete.m` is called again and the equations are further integrated in time. If the perturbation is instead in its asymptotic condition, the loop on the wavenumber range is incremented by one and the next wavenumber will be processed by `channel_main.m/wake_main.m`. When the last selected wavenumber, k_{fin}, verifies the asymptotic condition, the procedure stops.

3 Database setting

Here we describe the database structure. For the channel flow, the perturbative analysis considers 4 parameters (Reynolds number, symmetry/asymmetry, angle of obliquity and wavenumber). Every folder, corresponding to a certain specification of the above parameters, contains the following text files:

- `prefix_t_n.txt`: the temporal points, M, at which the solutions are computed through the Matlab code, are reported in column;

- `prefix_u_n.txt`, `prefix_v_n.txt`, `prefix_w_n.txt`: these files contain the perturbation velocity field components. Each of them has two columns, the first one for the real part and the second one for the imaginary part of the velocity component. The column length is MxN, where N is the number of spatial grid points ($N = 501$ for the channel flow) and M are the temporal instants. For every fixed time, the velocity spatial distributions are put in column;

- `prefix_omega_y_n.txt`: this file contains the transversal vorticity component and is structured as the above velocity field files;

- `prefix_energy_n.txt`: in this file, the kinetic energy density, e, is reported. The first column expresses the temporal points, M, and the second one the corresponding energy values.

Each of these files contains, as a prefix in its name, the parameter information and has an increasing number, n, as a suffix. This integer number, n, accounts for the fact that outputs are periodically saved (after a variable temporal interval) as the equations are integrated in time.

Let us make an example. Suppose we are interested in the simulation with parameters $Re = 500$, symmetric initial conditions, $\phi = \pi/4$, $k = 3$. If one goes to
`/Channel/Re_500/Re_500_sym/Re_500_sym_phi_45/Re_500_sym_phi_45_k_3`, he will find the above text files with prefix `Re_500_sym_phi_45_k_3` and suffix $n \geq 1$.

For the wake flow, data are organized in an analogous way to the one described for the channel flow. It should be recalled that 5 parameters are here considered (Reynolds number, wake position, symmetry/asymmetry, angle of obliquity and wavenumber). Therefore, in the corresponding folders and files, as part of the suffix, information on the chosen downstream station, x_0, are added similarly to what done for the other parameters. Moreover, concerning the perturbation velocity and the transversal vorticity files, it should be recalled that the number of grid points, N, depends on the wavenumber considered (see Section 2, $N = 801$ if $k > 1$, $N = 1201$ if $k \in [0.75, 1]$, $N = 1601$ if $k \in [0.45, 0.7]$).

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