EFMC-2006: The intermediate asymptotics of turbulent diffusion

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– M.Iovieno, D.Tordella 2002 "The angular momentum for a finite element of a fluid: A new representation and application to turbulent modeling", *Physics of Fluids*, **14**(8), 2673–2682.

– M.Iovieno, C.Cavazzoni, D.Tordella 2001 "A new technique for a parallel dealiased pseudospectral Navier-Stokes code." *Computer Physics Communications*, **141**, 365–374.

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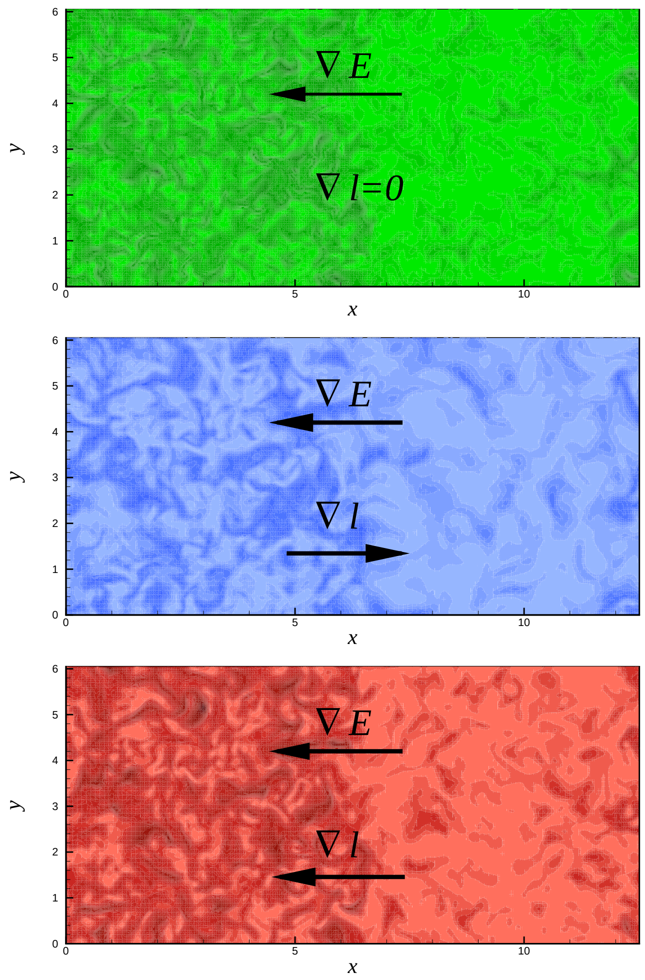
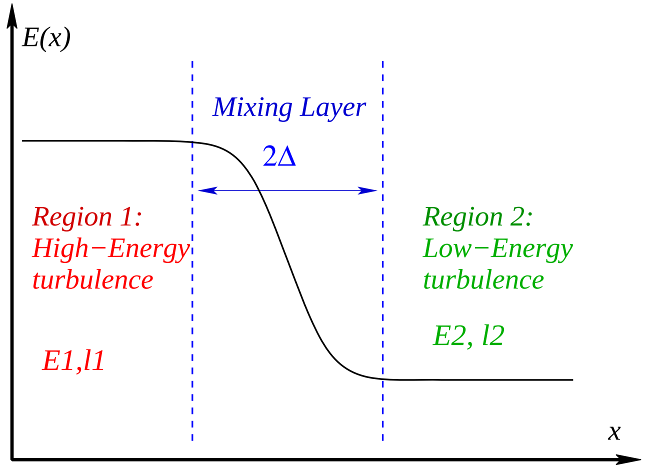
Shearless turbulence mixing: an overview





- mixing between homogeneous turbulences: no mean shear ⇒ *no turbulence production*
- the mixing layer is generated by the turbulence inhomogeneity, i.e.:
  - ◇ by the gradient of *turbulent energy*

  and
  - ◇ by the gradient of *integral scale*

Properties of laboratory/numerical numerical experimen-ts:

Gilbert, *JFM* **100** (1980), Veeravalli and Warhaft *JFM* **207** (1989) Briggs et al. *JFM* **310** (1996), Knaepen et al. *JFM* **414** (2004), Tordella and Iovieno *JFM* **549** (2006)

- A self-similar stage of decay is always reached
- It is characterized by a strong intermittent penetration, which depends on the two mixing parameters:
  - the turbulent energy gradient
  - the integral scale gradient

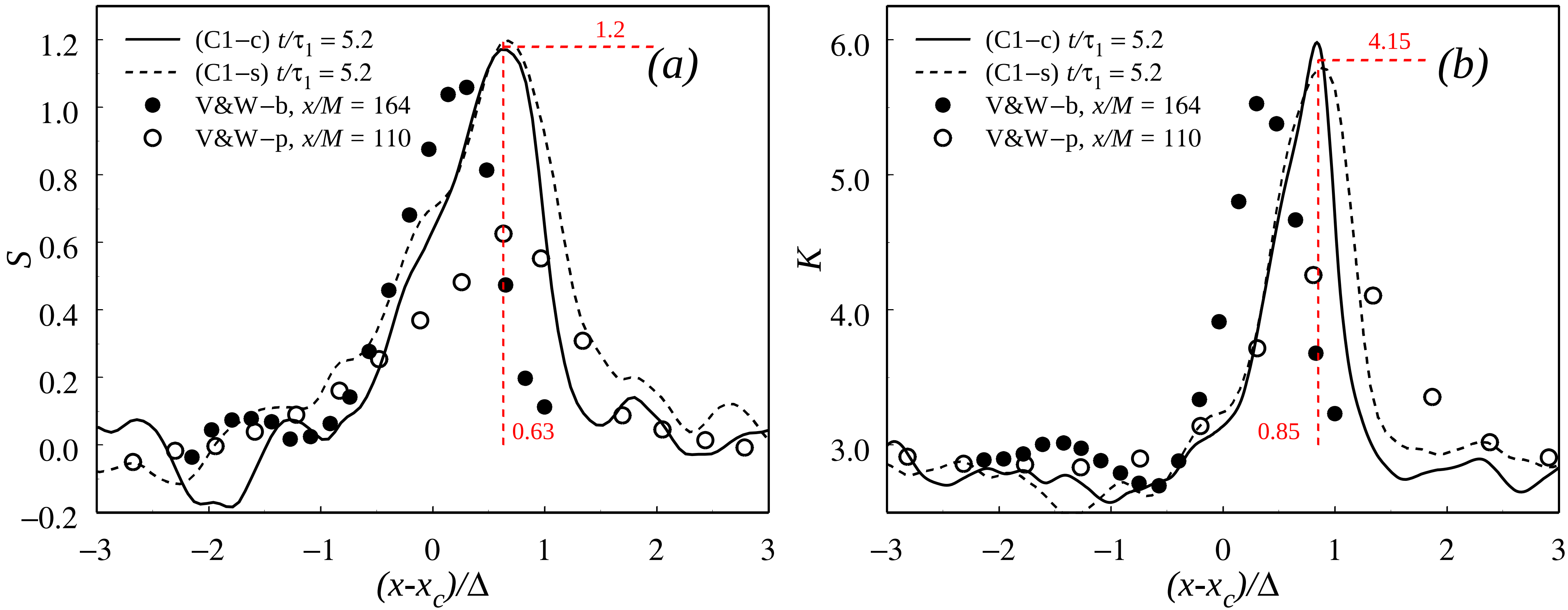
This behaviour must be contained in the solutions of:

- the two-point correlation equation which allows to consider both the macroscale and energy gradient parameters ($B_{ij}(\mathbf{x}, \mathbf{r}, t) = \overline{u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t)}$);
- the one-point correlation equation, the limit $\mathbf{r} \to \mathbf{0}$, which allows to obtain the third order moment (skewness) distribution from second order (energy) distribution.

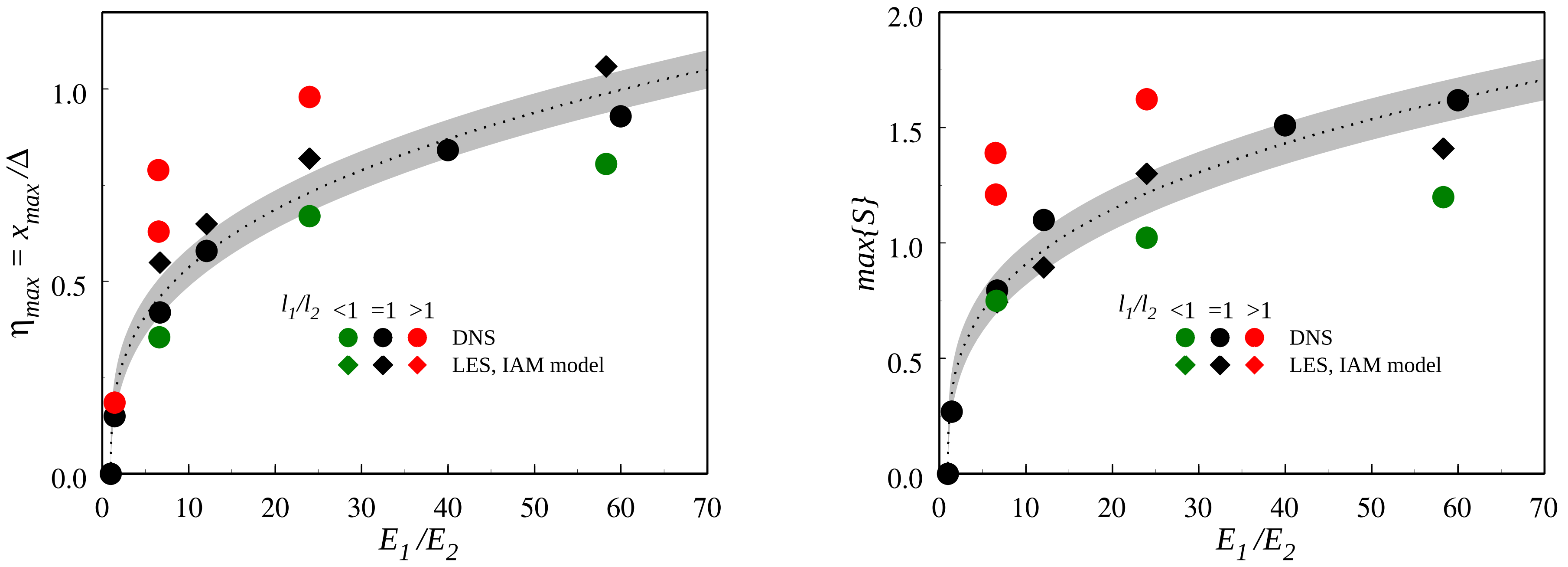
Intermittency: skewness and kurtosis profiles

$$S = \frac{\overline{u^3}}{\overline{u^2}^{\frac{3}{2}}} \quad K = \frac{\overline{u^4}}{\overline{u^2}^2} \;\Rightarrow\; S \approx 0, \quad K \approx 3 \text{ in homogeneous isotropic turb.}$$

$\mathcal{E} = 6.6, \mathcal{L} = 1.5$: gradients of energy and scale have the same direction.



Intermittent penetration η_{max} - maximum of skewness $max\{S\}$



for $\mathcal{L} = 1$ we have the approximate scaling law

$$\eta_{max} \sim a(\mathcal{E} - 1)^b, \quad a = 0.27, \; b = 0.33,$$

$$\max\{S\} \sim a(\mathcal{E} - 1)^b, \quad a = 0.46, \; b = 0.31$$

main reference: Tordella and Iovieno *JFM* **549** (2006)

Two-point double correlations:

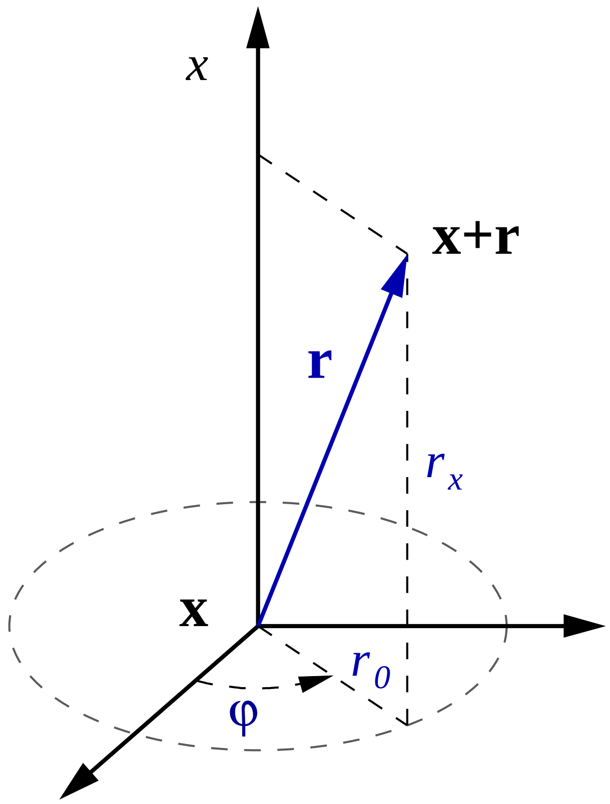
$$B_{ij}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)}$$
$$B_{pi}(\mathbf{x},\mathbf{r},t) = \overline{p(\mathbf{x},t)u_i(\mathbf{x}+\mathbf{r},t)}$$
$$B_{ip}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)p(\mathbf{x}+\mathbf{r},t)}$$

Two-point triple correlations:

$$B_{ij|k}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x},t)u_k(\mathbf{x}+\mathbf{r},t)}$$
$$B_{i|jk}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)u_k(\mathbf{x}+\mathbf{r},t)}$$

Momentum equations:

$$\frac{\partial}{\partial t}B_{ij}(\mathbf{x},\mathbf{r},t) + \frac{\partial}{\partial x_k}B_{ik|j}(\mathbf{x},\mathbf{r},t) + \frac{\partial}{\partial r_k}\Big(B_{i|kj}(\mathbf{x},\mathbf{r},t) - B_{ik|j}(\mathbf{x},\mathbf{r},t)\Big) =$$
$$= \frac{1}{\rho}\left\{-\frac{\partial}{\partial x_i}B_{pj}(\mathbf{x},\mathbf{r},t) + \frac{\partial}{\partial r_i}B_{pj}(\mathbf{x},\mathbf{r},t) - \frac{\partial}{\partial r_j}B_{ip}(\mathbf{x},\mathbf{r},t)\right\} +$$
$$+\nu\left[\frac{\partial^2}{\partial x_k \partial x_k}\frac{\partial}{\partial t}B_{ij}(\mathbf{x},\mathbf{r},t) + 2\frac{\partial^2}{\partial r_k \partial r_k}\frac{\partial}{\partial t}B_{ij}(\mathbf{x},\mathbf{r},t) - 2\frac{\partial^2}{\partial x_k \partial r_k}\frac{\partial}{\partial t}B_{ij}(\mathbf{x},\mathbf{r},t)\right]$$



Simmetries:

- there is only one non homogeneous and non isotropic direction, denoted by x
- correlations are invariant under all translations perpendicular to x and to all rotations around x

$\Rightarrow$ with cylindrical coordinates all variables are functions of (x, r_0, r_x, t) only

- $B_{\partial_x u_x \partial_x u_x}(x, r_0, 0, t) = \lambda^{-2}(x, r_0, t) B_{xx}(x, r_0, 0, t)$

Two-point lateral correlation B_{xx} equation for $r_x \to 0$ reduces to

$$\frac{\partial}{\partial t} B_{xx} + \frac{\partial}{\partial x} B_{xx|x} - 2\left(\frac{\partial B_{rx|x}}{\partial r_0} + \frac{B_{rx|x}}{r_0}\right) =$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial x} B_{px} + \nu \left\{ \left[\frac{\partial^2}{\partial x^2} + 2\left(\frac{\partial^2}{\partial r_0^2} + \frac{1}{r_0}\frac{\partial}{\partial r_0}\right)\right] B_{xx} - \frac{B_{xx}}{\lambda^2(x, r_0, t)} \right\} \quad (1)$$

Hypothesis and semplifications

- The two homogenous turbulences decay in the same way, thus

$$E_1(t) = A_1(t + t_0)^{-n_1}, \quad E_2(t) = A_2(t + t_0)^{-n_2}$$

the exponents n_1, n_2 are close each other (numerical experiments, Tordella & Iovieno, *JFM* 2006). Here, we suppose $n_1 = n_2 = n = 1$, a value which corresponds to $R_\lambda \gg 1$ (Batchelor & Townsend, 1948).
$\Rightarrow$ this implies $\mathcal{E} = E_1(t)/E_2(t) = \text{const}, \quad \mathcal{L} = \ell_1(t)/\ell_2(t) = \text{const}$

- Pressure-velocity correlation has been shown to be approximately proportional to the convective fluctuation transport (Yoshizawa, 1982, 2002) for $\mathbf{r} \longrightarrow 0$, so that

$$-\overline{pu_i} = a\rho\frac{\overline{u_i u_i u_j}}{2}, \quad a \approx 0.10$$

so that

$$-B_{px}(x, 0, 0, t) = 0.25\rho B_{xx|x}(x, 0, 0, t)$$

Similarity hypothesis

The two-point lateral correlations are determined by

- the coordinates x, r_0, t
- the energies $E_1(t)$, $E_2(t)$ $\Rightarrow$ gradient ∇E or their ratio $\mathcal{E} = E_1/E_2$
- the integral scales $\ell_1(t)$, $\ell_2(t)$ $\Rightarrow$ gradient $\nabla \ell$ or their ratio $\mathcal{L} = \ell_1/\ell_2$
- the mixing layer thickness $\Delta(t)$

The turbulent kinetic energy and the integral scale of the high energy region (E_1, ℓ_1), that is

$$B_{xx}(-\infty, 0, 0, t) = \frac{2}{3} E_1(t), \quad \ell_1(t)$$

are chosen as velocity and length scales.

Consequently all correlations can be expressed in terms of

$$\eta = \frac{x}{\Delta(t)}, \qquad \xi = \frac{r_0}{\ell(x,t)},$$

and this is the structure of the equations for the similarity analysis:

$$\begin{aligned}
B_{xx}(x, r_0, 0, t) &= B_{xx}(-\infty, 0, 0, t)\, \varphi_{xx}(\eta, \xi) \\
B_{xx|x}(x, r_0, 0, t) &= B_{xx}^{\frac{3}{2}}(-\infty, 0, 0, t)\, \varphi_{xx|x}(\eta, \xi) \\
B_{rx|x}(x, r_0, 0, t) &= B_{xx}^{\frac{3}{2}}(-\infty, 0, 0, t)\, \varphi_{rx|x}(\eta, \xi) \\
B_{px}(x, r_0, 0, t) &= \rho B_{xx}^{\frac{3}{2}}(-\infty, 0, 0, t)\, \varphi_{px}(\eta, \xi) \\
\lambda(x, r_0, t) &= \ell_1(t)\, \tilde{\lambda}(\eta, \xi) \\
\ell(x, t) &= \ell_1(t)\, \tilde{\ell}(\eta)
\end{aligned}$$

$\Rightarrow$ similarity conditions:

By introducing the similarity relations in the equation for B_{xx} and by imposing that all the coefficients must be independent from (x, r_0, t), the following condition is obtained

$$\Delta(t) \propto \ell_1(t) = \mathcal{L}\, \ell_2(t)$$

Equation (1) reduces to

$$-\varphi_{xx} + \frac{1}{2}\left[\eta \frac{\partial \varphi_{xx}}{\partial \eta} - \xi\eta\tilde{\ell}' \frac{\partial \varphi_{xx}}{\partial \xi}\right] + \frac{3}{4f(R_{\ell_1})}\left[\frac{\partial \varphi_{xx|x}}{\partial \eta} - \xi\tilde{\ell}' \frac{\partial \varphi_{xx|x}}{\partial \xi} - 2\frac{1}{\tilde{\ell}\xi}\frac{\partial}{\partial \xi}\left(\xi \varphi_{rx|x}\right)\right] =$$
$$= -\frac{3}{f(R_{\ell_1})}\left[\frac{\partial \varphi_{px}}{\partial \eta} - \xi\tilde{\ell}' \frac{\partial \varphi_{px}}{\partial \xi}\right] + \frac{3}{2f(R_{\ell_1})}\frac{1}{R_{\ell_1}}\left\{\left[\frac{\partial^2 \varphi_{xx}}{\partial \eta^2} - \xi\tilde{\ell}'\left(\frac{\partial^2 \varphi_{xx}}{\partial \eta \partial \xi} - \tilde{\ell}' \frac{\partial \varphi_{xx}}{\partial \xi} - \xi\tilde{\ell}' \frac{\partial^2 \varphi_{xx}}{\partial \xi^2}\right) - \xi\tilde{\ell}'' \frac{\partial \varphi_{xx}}{\partial \xi}\right]\right.$$
$$\left. + \left[\frac{2}{\tilde{\ell}^2 \xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \varphi_{xx}}{\partial \xi}\right) - \frac{\varphi_{xx}}{\tilde{\lambda}^2}\right]\right\}$$

boundary conditions: matching with the two homogeneous turbulences external to the mixing:

- for $\eta \rightarrow -\infty$ to homogeneous turbulence with energy E_1 and scale ℓ_1
- for $\eta \rightarrow +\infty$ to homogeneous turbulence with energy E_2 and scale ℓ_2

One-point limit

For $\xi \to 0$, this equation and its boundary conditions become

$$\frac{\partial \varphi_{xx|x}}{\partial \eta} = \frac{4 f(R_{\ell_1})}{3} \left\{ \frac{1}{2} \eta \frac{\partial \varphi_{xx}}{\partial \eta} + \frac{3}{2 f(R_{\ell_1})} \frac{1}{R_{\ell_1}} \frac{\partial^2 \varphi_{xx}}{\partial \eta^2} + \varphi_{xx} \left[1 - \frac{3}{f(R_\ell(\eta))} \frac{1}{R_\ell(\eta)} \frac{\tilde{\ell}^2(\eta)}{\tilde{\lambda}^2(\eta, 0)} \right] \right\}$$

where $R_\ell(\eta)$ is the *local* Reynolds number, i.e. based on local energy and scale

second order correlation b.c.:

$$\lim_{\eta \to -\infty} \varphi_{xx}(\eta, 0) = \frac{2}{3}, \quad \lim_{\eta \to +\infty} \varphi_{xx}(\eta, 0) = \frac{2}{3} \mathcal{E}^{-1},$$

third order correlation b.c.:

$$\lim_{\eta \to \pm\infty} \varphi_{xx|x}(\eta, 0) = 0$$

This equation relates the distributions of second and third order moments in the mixing layer.

The flow outside the mixing ($|\eta| \gg 1$) is homogeneous, thus

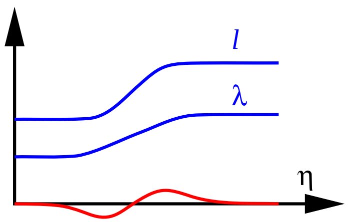
$$\frac{\partial}{\partial \eta} = 0 \quad \Rightarrow \quad \left[1 - \frac{3}{f(R_\ell(\eta))} \frac{1}{R_\ell(\eta)} \frac{\tilde{\ell}^2(\eta)}{\tilde{\lambda}^2(\eta, 0)}\right] = 0 \quad \text{for} \quad \eta \to \pm\infty \quad (\clubsuit)$$

this is consistent with homogeneous turbulence, where it is known that

$$\left(\frac{\ell}{\lambda}\right)^2 \propto R_\ell.$$

However, this is not necessarily true through the mixing layer:

- when $\mathcal{L} = 1$ there is still equilibrium as regards the integral scale, so we assume that this relation still holds
- when $\mathcal{L} \neq 1$ we suppose that function $(\clubsuit)$ is a function of the scale gradients, so that we could write



$$\tilde{\lambda}(\eta, 0) = \left(\frac{3}{f(R_\ell(\eta))R_\ell(\eta)}\right)^{\frac{1}{2}} \tilde{\ell}(\eta) \left(1 - \frac{b}{\varphi_{xx}} \frac{\mathrm{d}^2 \ell}{\mathrm{d}\eta^2}(\eta)\right)^{-\frac{1}{2}}$$

Now, the distribution of skewness can be obtained from the one-point correlation equation when a typical expression for the second order moment distribution is introduced as in Veeravalli & Warhaft *JFM* 1989:

$$\varphi_{xx}(\eta) = \frac{1+\mathcal{E}^{-1}}{2} - \frac{1-\mathcal{E}^{-1}}{2}\mathrm{erf}\,(\eta)$$

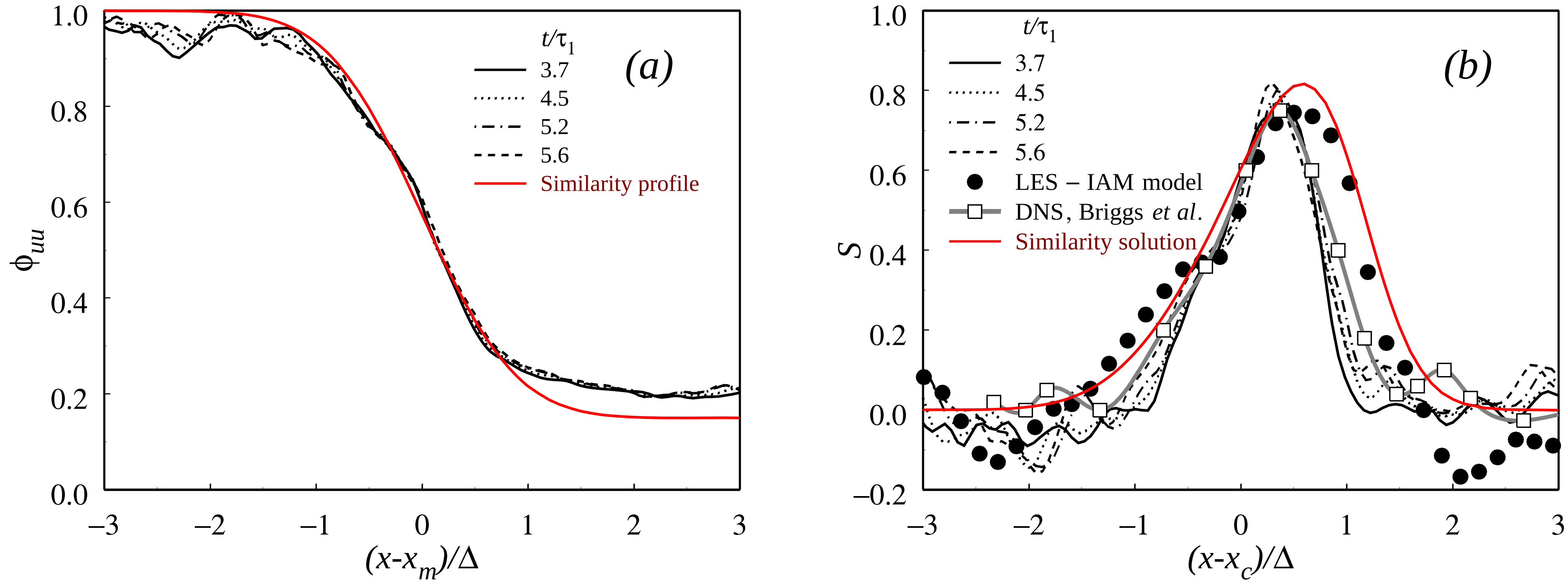
$$\tilde{\ell}(\eta) = \frac{1+\mathcal{L}^{-1}}{2} - \frac{1-\mathcal{L}^{-1}}{2}\mathrm{erf}\,(a\eta)$$

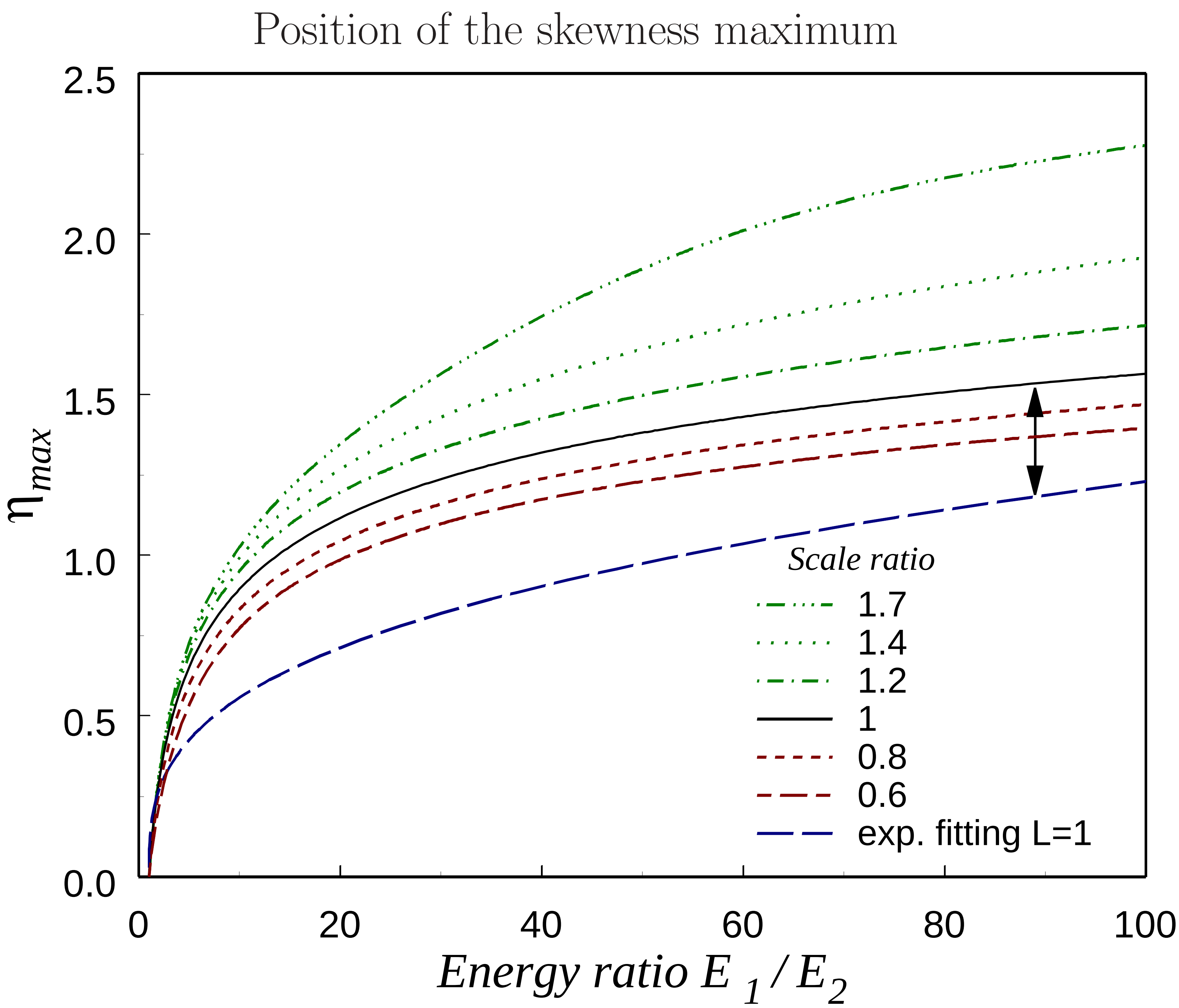
The third order moment is then

$$\varphi_{xxx} = \frac{f}{6}\frac{1-\mathcal{E}^{-1}}{\sqrt{\pi}}\left[\left(1-\frac{6}{f(R_{\ell_1})}\right)\mathrm{e}^{-\eta^2} + ba(1-\mathcal{L}^{-1})\mathrm{e}^{-(a\eta)^2}\right].$$

Parameters $a > 0$ and $b > 0$ are chosen to fit the experimental behaviour

Normalized energy and skewness distributions: $\mathcal{E} = 6.7$ and $\mathcal{L} = 1$.





Position of the skewness maximum
2.5
2.0
1.5
1.0
0.5
0.0
η_{max}
0
20
40
60
80
100
Energy ratio E_1 / E_2
Scale ratio
1.7
1.4
1.2
1
0.8
0.6
exp. fitting L=1

Conclusions

The intermediate asymptotics of the turbulence diffusion in the absence of production of turbulent kinetic energy is considered.

- An intermediate similarity stage of decay always exists.
- When the energy ratio $\mathcal{E}$ is far from unity, the mixing is very intermittent.
- when $\mathcal{L} = 1$, the intermittency increases with the energy ratio $\mathcal{E}$ with a scaling exponent that is almost equal to 0.3.
- intermittency smoothly varies when passing through $\mathcal{L} = 1$:
  it increases when $\mathcal{L} > 1$ (*concordant* gradient of energy and scale),
  it reduces when $\mathcal{L} < 1$ (*opposite* gradient of energy and scale)
- the similarity solution contains all the salient points showed by the experiments.

→ *Animations (kinetic energy):* $\mathcal{L} = 0.6$ $\mathcal{L} = 2.1$

Similarity equation

$$
-\varphi_{xx} + \frac{1}{2}\left[\eta\frac{\partial\varphi_{xx}}{\partial\eta} - \xi\eta\tilde{\ell}'\frac{\partial\varphi_{xx}}{\partial\xi}\right] +
$$

$$
+ \frac{3}{4f(R_{\ell_1})}\left[\frac{\partial\varphi_{xx|x}}{\partial\eta} - \xi\tilde{\ell}'\frac{\partial\varphi_{xx|x}}{\partial\xi} - 2\frac{1}{\tilde{\ell}\xi}\frac{\partial}{\partial\xi}\left(\xi\varphi_{rx|x}\right)\right] =
$$

$$
= -\frac{3}{f(R_{\ell_1})}\left[\frac{\partial\varphi_{px}}{\partial\eta} - \xi\tilde{\ell}'\frac{\partial\varphi_{px}}{\partial\xi}\right] +
$$

$$
+ \frac{3}{2f(R_{\ell_1})}\frac{1}{R_{\ell_1}}\left\{\left[\frac{\partial^2\varphi_{xx}}{\partial\eta^2} - \xi\tilde{\ell}'\left(\frac{\partial^2\varphi_{xx}}{\partial\eta\partial\xi} - \tilde{\ell}'\frac{\partial\varphi_{xx}}{\partial\xi} - \xi\tilde{\ell}'\frac{\partial^2\varphi_{xx}}{\partial\xi^2}\right) - \xi\tilde{\ell}''\frac{\partial\varphi_{xx}}{\partial\xi}\right]\right.
$$

$$
\left. + \left[\frac{2}{\tilde{\ell}^2\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\varphi_{xx}}{\partial\xi}\right) - \frac{\varphi_{xx}}{\tilde{\lambda}^2}\right]\right\}
$$