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A new matched asymptotic expansion for the intermediate and far flow behind a finite body

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An approximated Navier–Stokes steady solution is here presented for the two dimensional bluff body wake region that is intermediate between the field on the body scale L_D, which includes the two symmetric counter-rotating eddies, and the ultimate far wake. The nonparallelism of the streamlines in the intermediate wake cannot yet be considered negligible. The R is of the order of the critical value for the onset of the first instability and the limiting behavior for large R is not considered. The solution is obtained by matching an inner solution—a Navier–Stokes expansion in powers of the inverse of the longitudinal coordinate—and an outer solution, which is a Navier–Stokes asymptotic expansion in powers of the inverse of the distance from the body. The matching is built on the criteria that, where the two solutions meet, the longitunal pressure gradients and the vorticities must be equal and the flow toward the inner layer must be equal to the outflow from the external stream. At high orders in the inner expansion solution, the lateral decay turns out to be algebraic. This approximate solution is here examined in relation to the class of asymptotic solutions that, in the past, were obtained by adopting the rapid decay principle, which implies an irrotational outer flow. The theme running through this paper is the necessity of the addition of this criterion to the equations of motion to build a solution that describes the intermediate wake. The present solution has been obtained by relaxing the imposition of the rapid decay principle. It can be concluded that, at Reynolds numbers as low as the first critical value and where the nonparallelism of the streamlines is not yet negligible, the division of the field into two basic parts—an inner vortical boundary layer flow and an outer potential flow—is spontaneously shown up to the second order of accuracy: at higher orders in the expansion solution the vorticity is first convected and then diffused in the outer field. If exploited to represent the basic flow of bluff body wakes, the analytical simplicity of this asymptotic expansion could be useful for the nonparallel analysis of the instability of two-dimensional wakes. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1580482]

I. INTRODUCTION

To analyze the nonparallel effects on the stability of two-dimensional (2-D) wakes it could be advantageous to have an analytical description of the basic flow that is more accurate than the famous far field Gaussian asymptotic representation[1] and which is available in the intermediate wake region at Reynolds numbers around the first critical value.

In this paper a simple nonparallel Navier–Stokes asymptotic expansion is proposed for the intermediate and far wake, see Fig. 1. Apart from describing the streamwise momentum distribution, this expansion also describes the transversal momentum (hence the streamline curvature) and pressure distributions and is valid at finite values of the Reynolds number of the order of the critical value for the onset of the first instability, $R\sim 20-50$. The term "intermediate" is used in the general sense as given by Zeldovich and Barenblatt[2] (cf. the Preface, page xiii). The recirculating region,[3–5] which, at relatively low R, ranges on the body scale and which, up to $R=160$, extends to dimensions $L=O(RD)\times W=O(R^{1/2}D)$ ($L=$ length, $W=$width), is not included in this expansion. The idea that the solution proposed here could be used as an accurate basic field for the study of the *nonparallel* linear instability of 2-D wakes is supported by the remarkable numerical experiment by Triantafyllou and Karniadakis,[6] which proves that the details of the flow separation from the body that generates the wake can be disregarded in wake-stability analyses.[7] The steady wake limiting behavior for $R\to\infty$ (see, e.g., Fornberg,[8] Chernyshenko,[9,10] Peregrine[11]) is not considered in this study either.

According to the Oseen type of successive approximations and with the adoption of the rapid decay principle, a number of truncated expansion solutions were found.[12–16] However, in the literature concerning wake instability, these expansions were never used as basic flows to be perturbed, presumably because of their analytical complexity due to the presence of logarithmic terms, which had to be included to maintain the exponential nature of the lateral decay.

On the other hand, it should be recalled that rapid decay has never been demonstrated even for the far wake, which explains why it is used as (and called) a principle, see, for

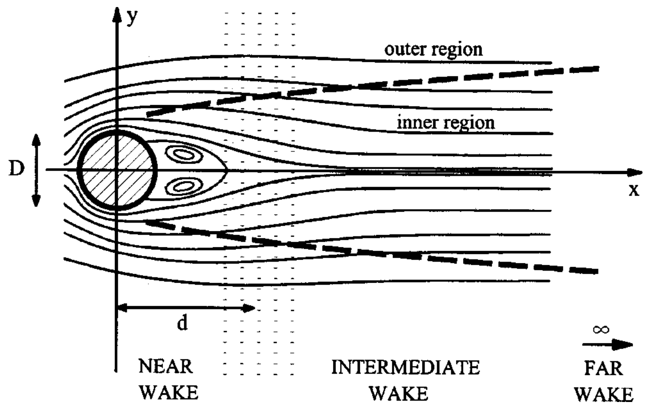

FIG. 1. Sketch of the regions of the laminar wake flow behind a 2-D bluff body.

example, Stewartson (Ref. 13, p. 177), Chang (Ref. 15, p. 834), and Kida (Ref. 16, p. 949). At the same time it is also interesting to notice that similarity solutions, in which the vorticity decays algebraically at the edge of the viscous inner layer, were shown to be possible limit solutions of full boundary-layer equations with exponential decay associated with a potential outer flow.[17]

In view of the fact that, at low Reynolds number and finite distances from the bluff body, a full Navier–Stokes solution is a more acceptable outer flow model, from a physical point of view, than a potential solution, here it was decided to generalize the modeling of the inner and outer layers and to free the analysis from the addition of any decay conditions. A matching of two asymptotic Navier–Stokes expansion solutions was sought for both the inner and the outer layers at fixed Reynolds numbers. The inner layer solution (Sec. II B) was built in terms of a near similarity expansion in powers of the inverse longitudinal coordinate, see Belan and Tordella,[18] where, in the framework of the boundary layer model, an analog expansion solution was found up to any order of accuracy. The outer layer (Sec. II C) was built in terms of powers of the inverse of the distance from the bluff body that shapes the wake.

As the intermediate (x finite) steady two-dimensional wake at low Reynolds numbers ($R \lesssim 40$) is a system in which the dynamics consists of the transport, through nonlinear convection, and of the diffusion of vorticity, this latter quantity was chosen, rather than the velocity, as the physical quantity on which to base the process of matching. The longitudinal pressure gradients generated by the flow and the entrainment velocities are also matched, see Sec. II A. The pressure effects in the inner layer have been considered of relevance whenever the transversal momentum balance shows a pressure term of a magnitude that can be compared with the diffusion and convection terms.

The synthetic list of the properties of the wake flow which have here been taken into account and which were not taken into account in previous literature is presented in Sec. II. The terms of the expansion solution are presented up to the fourth order in Sec. III. A comparison between Chang's exponentially decaying asymptotic expansion and the present Navier–Stokes expansion solution, as applied to the flow past the circular cylinder, is presented in Sec. IV, together with a comparison of the experimental laboratory distributions by Kovasznay[19] and numerical distributions by Berrone.[20,21]

The present investigation supports the argument that solutions with algebraic lateral decay play a role that is only apparently antithetical to that played by the exponentially decaying expansions. It is in fact complementary, as it is relevant to a more extended portion of the wake field that includes, apart from the far region where the decay becomes asymptotically exponential, the intermediate region where the Oseen approximation loses accuracy.

It should be recalled that, for the Navier–Stokes model in an exterior unbounded 2-D domain at Reynolds numbers as high as the critical value for the onset of the first instability, properties of existence and uniqueness of the solution have not yet been demonstrated, see the monographs by Galdi (Ref. 22, Vol. II) and Ladyzhenskaya (Ref. 23).

II. BASIC EQUATIONS AND THE PHYSICAL PROBLEM

For the incompressible viscous flow past a bluff body, the nondimensional continuity and Navier–Stokes equations are written in the form

$$u\,\partial_x u + v\,\partial_y u + \partial_x p - R^{-1}\,\nabla^2 u = 0, \tag{1}$$

$$u\,\partial_x v + v\,\partial_y v + \partial_y p - R^{-1}\,\nabla^2 v = 0, \tag{2}$$

$$\partial_x u + \partial_y v = 0, \tag{3}$$

where (x,y) are the longitudinal and normal coordinates, (u,v) the component velocities, p the pressure, and $R \in [10, R_{cr} \sim 40]$ the Reynolds number. The adopted adimensionalization is based on the characteristic length of the flow (a typical length D of the body that generates the wake), the density ρ, and the velocity U of the free stream. Both the outer and the inner flows are required to satisfy this model, while no linearization is carried out. The specification of the problem is then completed with the system of boundary conditions which excludes the portion of flow on the body scale and involves symmetry to the longitudinal coordinate and uniformity at infinity. Furthermore, there is a body of experimental knowledge that offers a particularly rich description of the inner layer, which needs to be inserted into the relevant boundary and matching conditions. For this purpose, the inner flow is required (i) to be a thin layer described by the Navier–Stokes model, (ii) to match its momentum defect, with respect to the incoming stream, to the body drag, i.e., to keep its momentum constant along the x direction, and iii) to entrain external fluid,[24] see the detailed presentation of accessory conditions in the following section. The outer flow is considered as a Navier–Stokes flow which symmetrically wraps a thus characterized inner flow and at the same time accommodates $u \to U, v \to 0, p \to p_\infty$ for $y \to \pm\infty$.

The considered domain is composed of the intermediate and far wake

$$d < x < \infty; \quad -\infty < y < \infty, \tag{4}$$

where x is the standard longitudinal coordinate—with the origin placed in the center of the body that generates the wake—and $d>0$ is the distance, which decreases with R, from the center of the body beyond which the thin shear layer model becomes relevant. Function d, nominally a function of both the R and the shape of the body, is a free parameter. Its value, at fixed R, can in theory be obtained by means of the matching with the pre-asymptotic flow. It seems reasonable to assume that the intermediate flow region begins at $x=d$. Thus, according to the definition of intermediate asymptotics, d should not depend to any great extent on the details of the actual shape. Distance d usually varies from eight to four diameters for $R\in[20,40]$).[18,19] Both the origin and the near wake, which includes the symmetrical adherent vortices, fall outside the domain of our analysis. As a consequence it is necessary to introduce field information that gives one of the accessory conditions along the x coordinate, as suggested by Stewartson,[13] i.e., the profiles

$$
\begin{aligned}
u(x_*,y;R)&=u_*(y;R),\\
v(x_*,y;R)&=v_*(y;R),\\
p(x_*,y;R)&=p_*(y;R)
\end{aligned}
\tag{5}
$$

of an experimental nature, which are both the result of a numerical simulation and of a laboratory measurement, placed in the intermediate field at $x=x_*$. The second condition along x is the uniformity condition at infinity.

At this point, it is opportune to summarize the differences that characterize the present approach with respect to the previous literature. (i) The recognition of the existence of the *intermediate asymptotics*. This is a very important point, as the existence of the intermediate region physically introduces the adoption of the thin shear layer hypothesis, and relevant near-similar variable transformations for the inner flow, while, at the same time, it supports a differentiation of the behavior of the intermediate flow with respect to its infinite asymptotics. (ii) The use of the *in-field* boundary condition (5) which has a higher degree of field information than the mere use of integral quantities such as the drag or the lift coefficients, which however are in turn included in (5). (iii) The adoption of the *inner* as a basic approximation, which means that, up to first order, the inner solution is independent of the outer solution. Coherently with this matching order, the Navier–Stokes model, coupled with the thin layer hypothesis, very naturally yields the order of the field pressure variations $O(x^{-2})$, see Sec. II B. The pressure variations were usually overestimated at $O(x^{-1})$ in previous studies,[15,16] see also Sec. IV. (iv) The use of the Navier–Stokes equations in all the considered field, *without the addition* of further restrictive axiomatic positions such as the principle of exponential decay. This does not prevent the present solution from showing the properties of rapid decay and irrotationality at first and second order for the inner and the outer flows, respectively. At the higher orders, which mainly influence the intermediate region, the decay becomes a fast algebraic decay. See Secs. III–V.

A. Matching rules and structure of the expansion solution

The matching on the pressure forces is not performed directly on the pressure, but on its gradient, which is the actual quantity that the equations of incompressible motion control. As the pressure is only differentiated once along the coordinates, only one condition can be considered. In order to take into consideration that the flow nonparallelism implies a streamwise evolution of the field, we can impose

$$\lim_{y\to 0}\partial_x p_o=\lim_{y\to\infty}\partial_x p_i \quad \text{for } x \text{ fixed,} \tag{6}$$

where the subscripts indicate outer and inner variables, respectively. Since the wake dynamics is mainly a balance between the convection and the diffusion of the vorticity which is generated at the body surface, it is considered physically more significant to impose that the matching is on the vorticity rather than on the velocity. In this manner restrictive conditions of irrotationality are not imposed on the outer flow while, at the same time, an irrotational configuration is not *a priori* excluded for the outer flow. Hence for x fixed

$$\lim_{y\to 0}\omega_o=\lim_{y\to\infty}\omega_i. \tag{7}$$

To take the entrainment into account, it is necessary to match the outer and inner values of v at the transition between the outer and inner fields, which yields

$$\lim_{y\to 0}v_o=\lim_{y\to\infty}v_i \quad \text{for } x \text{ fixed.} \tag{8}$$

This set of simple matching rules is applied in the following whenever possible, when the limit values are both finite. On some occasions these rules are improved by using the limiting behavior of the quantities being matched, which are written as asymptotic expansions in the primitive independent variables x,y[25] (Sec. III).

The structure of the inner and outer expansion solutions is sought in the class of inverse coordinate expansions that satisfies the boundary conditions at infinity and allows a partial variable separation which leads to a sequence of linear systems of ordinary inhomogeneous differential equations for the two groups of dependent variables $(u_i,v_i,p_i),(u_o,v_o,p_o)$. For the inner layer, the quasi-similar transformation is introduced,

$$\xi=x,\ \eta=x^{-1/2}y, \tag{9}$$

which assures the thinness of the inner domain. The introduction of the expansion hypothesis

$$f_i=f_{i0}(\eta)+x^{-1/2}f_{i1}(\eta)+x^{-1}f_{i2}(\eta)+\cdots \tag{10}$$

for the inner variables therefore allows the condition at $x\to\infty$ to be satisfied and, at the same time, the resulting inhomogeneous differential system $\vec{\mathcal{I}}_n(f_{in},\eta,\partial_\eta,\partial_\eta^2)=\vec{\mathcal{J}}_n(f_{i0},\ldots,f_{i(n-1)},\eta,\partial_\eta,\partial_\eta^2;R)$, obtained by introducing (9) and (10) into Eqs. (1)–(3), to be linear at each order. This is possible because, at each order, the variable separation implied by (10), though partial, leads the nonlinear terms in (1) to include only the products of quantities of an order of less than n, and these eventually end up in the inhomogeneous term. It should be noted that the expansion hypothesis (10), which fixes the functional dependence of the inner variables on x, actually makes the second relationship in (5) useless, because, once $u_i(x,y)$ is known, $v_i(x,y)$ is obtained by continuity. This is positive, since the experimental v profiles suffer from the inaccuracy that is associated with the smallness of values relevant to a quantity which is usually much lower than u. The quasi-similarity is due to the fact that while each single term of (10) is self-similar, their sum is not.

For the outer flow we introduce the variable transformation

$$r=(x^2+y^2)^{1/2}, \quad s=y/x \tag{11}$$

and the expansion hypothesis for the three (u_o, v_o, p_o) dependent variables

$$f_o=f_{o0}(s)+r^{-1/2}f_{o1}(s)+r^{-1}f_{o2}(s)+\cdots \tag{12}$$

which satisfy the asymptotic outer conditions at infinity. If (11) and (12) are introduced into (1)–(3), both the nonlinear and the diffusive terms include only quantities of orders of less than $n-1$ at each order. Thus, all the nonlinear and diffusion effects are confined to the inhomogeneous terms, a fact that reduces the differential order of the transformed equations by one and makes them linear. The new system is therefore an inhomogeneous linear ordinary differential system of the third order of the form: $\vec{\mathcal{O}}_n(f_{on}, s, \partial_s) = \vec{\mathcal{P}}_n(f_{o0}, \ldots, f_{o(n-1)}, s, \partial_s; R)$. The order of the inner system sums up to four, and as a consequence four constants of integration are introduced at each order. Two of these can be determined through symmetry requirements. The outer system contains three constants of integration at each order. The latter constants, together with the two integration constants obtained from the inner layer, are determined through the field boundary condition (5) (fitted by the least squares method)—which is actually a double condition on the variables u and p, since the v profile, according to the previous discussion, is unnecessary—and the three matching conditions (6)–(8). This set of conditions specifies the vectorial application: $\vec{\mathcal{M}}_n:[\mathcal{C}_i]_n \leftrightarrow [\mathcal{C}_o]_n$ that links the constants of integration at each order.

B. Inner expansion

The inner expansion is defined in the region where

$$x>d(R) \quad , \quad |y|\lesssim \mathcal{Y}(x) \quad \Rightarrow \quad \frac{y}{x}\to 0 \quad \text{as} \quad x\to\infty, y\to\infty \, , \tag{13}$$

where $\mathcal{Y}(x)$ is a representation of the boundary between the inner and the outer regions, which could almost be regarded as a parabola of the kind $\mathcal{Y}^2 \propto x$, but which can only be determined *a posteriori*. An inner expansion solution of the Navier–Stokes equations (NS in what follows, NSx equation along the x coordinate, NSy equation along the y coordinate) is defined for the wake region according to hypothesis (10).[18] The coordinate transformation (9) is here rewritten together with the relevant spatial derivative tranformations:

$$\begin{aligned} &\xi=x, \quad \partial_x=\partial_\xi-\tfrac{1}{2}\xi^{-1}\eta\partial_\eta, \\ &\eta=x^{-1/2}y, \quad \partial_y=\xi^{-1/2}\partial_\eta. \end{aligned} \tag{14}$$

According to (10), renaming $\xi\to x$, the velocity and pressure expansions are

$$\begin{aligned} u_i&=\phi_0(\eta)+x^{-1/2}\phi_1(\eta)+x^{-1}\phi_2(\eta)+\cdots, \\ v_i&=\chi_0(\eta)+x^{-1/2}\chi_1(\eta)+x^{-1}\chi_2(\eta)+\cdots, \\ p_i&=\pi_0(\eta)+x^{-1/2}\pi_1(\eta)+x^{-1}\pi_2(\eta)+\cdots. \end{aligned} \tag{15}$$

Some preliminary considerations must be introduced at this point. The structure of this expansion is such that continuity assures $\chi_0\equiv 0$. This fact is confirmed by the uniformity condition at $x\to\infty$, which also determines the other two function cofficients at order zero: $\phi_0(\eta)=1$, $\pi_0(\eta)=p_\infty/\rho U^2$. From continuity it can also be verified that $\chi_1(\eta)\equiv 0$. Thus the velocity component $v_i=x^{-1}\chi_2(\eta)+\cdots\approx O(x^{-1})$. As a general rule coefficients χ_n may be obtained directly from the continuity equation through the coefficients ϕ_{n-1}.

By substituting the change of coordinates (14) and the expansion form (15) in the NSx equation, a general ordinary differential equation for ϕ_n, $n\geqslant 1$, is obtained:

$$\mathcal{L}_n\,\phi_n \equiv \frac{1}{R}\phi_n''+\frac{\eta}{2}\phi_n'+\frac{n}{2}\phi_n=M_n, \tag{16}$$

where the inhomogeneous term M_n is made up of three parts:

$$M_n=T_n+P_{g\,n}+S_{d\,n}. \tag{17}$$

The first one, T_n, comes from the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in the NSx equation. It can be seen that $T_0=T_1=0$, $T_2=-\frac{1}{2}\phi_1^2$ and for $n\geqslant 3$,

$$T_n=-\frac{n}{4}\sum_{i=1}^{n-1}\phi_i\phi_{n-i}+\sum_{i=1}^{n-2}\left(-\frac{\eta}{2}\phi_i'\phi_{n-i}+\phi_i'\chi_{n-i}\right). \tag{18}$$

The terms $P_{g\,n}$ and $S_{d\,n}$ correspond to the pressure gradient component $\partial_x p$ and the streamwise diffusion term $\partial_x^2 u/R$, respectively. Their analytic expression can be deduced by substitution of expansions (15) in the NSx equation. Both these terms become identically equal to zero at any order in the simpler boundary layer model, where they may be considered as high order Navier–Stokes corrections.[18] Beginning from this fact, the pressure variations have been considered to be effectively present in the field starting from the order of accuracy, which assures the presence in the NSy equation of a pressure term that is comparable with at least one of the convective and diffusive terms. On inspection, assuming $u_i=1+x^{-1/2}\phi_1+\cdots$ and $v_i=x^{-1}\chi_2(\eta)+\cdots$, the NS$y$ equation shows that $n=3$ is the lowest order, which leads to a transversal pressure gradient of the same order as the convective (longitudinal) and diffusive (transversal) terms. Thus it may be supposed that $p_i=\pi_0+x^{-3/2}\pi_3(\eta)+O(x^{-2})$, i.e., $\partial_y p_i\approx O(x^{-2})$. However, a check in the NSy, written up to orders leading over $O(x^{-5/2})$, yields $\pi_3'(\eta)=x^{-3/2}((1/R)\chi_2''+=(\eta/2)\chi_2'+\chi_2)+O(x^{-2})$. *A posteriori* it is found that $(1/R)\chi_2''+(\eta/2)\chi_2'+\chi_2\equiv 0$, which yields $p_i=\pi_0+x^{-2}\pi_4(\eta)+O(x^{-5/2}), \pi_1=\pi_2=\pi_3=0$.

A comment is now opportune. As far as the pressure effects are concerned, an alternative position could have been to suppose that $p_i=\pi_0+x^{-1/2}\pi_1(\eta)+\cdots$. In the case in which the NSy yields $\pi_1'=\pi_2'=0$ and $\pi_3'(\eta)=x^{-3/2}((1/R)\chi_2''+(\eta/2)\chi_2'+\chi_2)+O(x^{-2})$, i.e., $\pi_1=K_1, \pi_2=K_2$ and $\pi_3(\eta)=K_3+\int((1/R)\chi_2''+(\eta/2\rangle\chi_2'+\chi_2)d\eta$, where now $\chi_2=\chi_2(\eta;K_1)$. This position thus leads to $p_i=\pi_0+K_1x^{-1/2}+K_2x^{-1}+\pi_3x^{-3/2}+O(x^{-2})$. From this one can infer that a field exists where, at the leading orders, the pressure varies conspicuously along the x direction, while it is constant along the normal direction. The significance of this scenario is considered physically questionable, because in an unconfined wake—a flow where no pressure variation is introduced from the outside—the outer field should not be able to impose strong longitudinal variations, inasmuch as it is simply the portion of the field where matching with the uniformity at infinity is obtained. Moreover, due to the concomitant presence of a constant inhomogeneous term in the differential equations for the coefficients ϕ_1 and ϕ_2, this pressure behavior would induce an anomalous plateau in the central field of the outer velocity, which in turn induces anomalous high values in the central and overshoot regions of the combined velocity field, which have not been experimentally observed,[5,19] see Fig. 5. The treatment adopted for the inner pressure field yields $P_{g\,n}=0$ for $n=1,2,3$ ($P_{g\,0}=0$ for the condition at infinity).

Independent of this, the streamwise diffusion $S_{d\,0}=S_{d\,1}=S_{d\,2}=0$, $S_{d\,3}=(4R)^{-1}(3\phi_1+5\eta\phi_1'+\eta^2\phi_1'')$. For $n\geqslant 4$, both the $S_{d\,n}$ and $P_{g\,n}$ terms are nonzero and it is possible to write them as functions of $\phi_0,\ldots,\phi_{n-1}$ together with their derivatives, which are quantities all known at the previous orders. As a result the equations of motion are converted into a hierarchy of ordinary differential systems, which can be written as

$$\begin{aligned}
&\phi_0'=0, \quad \mathcal{L}_n\,\phi_n=M_n, \quad n\geqslant 1,\\
&\chi_0'=0, \quad \chi_n'=\frac{\eta}{2}\phi_{n-1}'+\frac{n-1}{2}\phi_{n-1}, \quad n\geqslant 1,\\
&\pi_n'=0, \quad n=0,...,3;\\
&\pi_n'=\Pi_n(\phi_0,\ldots,\phi_{n-1},\chi_0,\ldots,\chi_{n-1}), \quad n\geqslant 4,
\end{aligned} \tag{19}$$

where as previously seen, $\phi_0=1$, $\chi_0=0$, $\pi_0=p_\infty/(\rho U^2)$, $\pi_1=\pi_2=\pi_3=0$.

The first equation can be solved directly for ϕ_n,[18] leading to

$$\phi_n(\eta)=A^n e^{-(R/4)\eta^2}\left[C_{n\,1}F_1\left(\frac{1-n}{2},\frac{1}{2};\frac{R}{4}\eta^2\right) + R\,\mathrm{Hr}_{n-1}(\eta)F_n(\eta)\right], \tag{20}$$

where A is a factorization of the C_n integration constants (other constants are determined by the symmetry and boundary conditions at infinity), ${}_1F_1$ is the confluent hypergeometric function (Ref. 26, Vol. 1, pp. 427, 473, 475) functions $\mathrm{Hr}_{n-1}(\eta)=H_{n-1}(\frac{1}{2}R^{1/2}\eta)$, where H_n are Hermite polynomials, and

$$F_n(\eta)=\int \frac{e^{(R/4)\eta^2}}{\mathrm{Hr}_{n-1}^2(\eta)}G_n(\eta)\,d\eta, \tag{21}$$

$$G_n(\eta)=A^{-n}\int M_n(\eta)\mathrm{Hr}_{n-1}(\eta)\,d\eta. \tag{22}$$

For $n\geqslant 3$, these integrals can be evaluated numerically or approximated using special functions. Once ϕ_n is known, the second equation in (19) gives

$$\chi_n=\frac{\eta}{2}\phi_{n-1}+\frac{n-2}{2}\Phi_{n-1} \tag{23}$$

with $\Phi_n=\int_0^\eta\phi_n(\zeta)d\zeta$, where the constant of integration was determined by symmetry. The π_n are obtained by direct integration of the relevant equation in (19). The order $n=0$ does not foresee any dependence on x, the pressure is constant, and the relevant integration constant K_0 is settled by the boundary condition at infinity. Since the field variations of p_i start to appear at the fourth order, the integration constants K_1, K_2, K_3 are set to zero.

C. Outer expansion

The outer expansion is defined in the region behind the body and outside the wake, i.e., the region where

$$x>d(R), \quad |y|\gtrsim\mathcal{Y}(x)\Rightarrow\frac{y}{x}\rightarrow\text{const}\neq 0 \quad \text{as} \quad x\rightarrow\infty, y\rightarrow\infty. \tag{24}$$

Since the left boundary of the whole domain lies behind the body, the outer region is made up of two symmetrically unconnected parts.

The adopted outer coordinate transformation (11) is here rewritten, together with the relevant spatial derivative transformations:

$$\begin{aligned}
&r=(x^2+y^2)^{1/2}, \quad \partial_x=s_-\partial_r-(s\,s_+/r)\partial_s,\\
&s=y/x, \quad \partial_y=s\,s_-\partial_r+(s_+/r)\partial_s,
\end{aligned} \tag{25}$$

where

$$s_\pm=(1+s^2)^{\pm 1/2}. \tag{26}$$

According to hypothesis (12), the velocity and pressure expansions are

$$\begin{aligned}
&u_o=u_0(s)+r^{-1/2}u_1(s)+r^{-1}u_2(s)+\cdots,\\
&v_o=v_0(s)+r^{-1/2}v_1(s)+r^{-1}v_2(s)+\cdots,\\
&p_o=p_0(s)+r^{-1/2}p_1(s)+r^{-1}p_2(s)+\cdots.
\end{aligned} \tag{27}$$

By substituting in the NS equations, together with the continuity equation, a hierarchy of ordinary differential systems is obtained. The general system of order n can always be rearranged as

$$\begin{aligned}
&u_n'=-\frac{n}{2}s_-^2(u_n/s+v_n+p_n/s)+U_n,\\
&v_n'=-\frac{n}{2}s_-^2p_n+V_n,\\
&p_n'=\frac{n}{2}s_-^2v_n+P_n,
\end{aligned} \tag{28}$$

where U_n, V_n, P_n are other inhomogeneous terms, made up of nonlinear combinations of $u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}, \pi_0, \ldots, \pi_{n-1}$ together with their derivatives. These terms correspond to the nonlinear and diffusive terms of the original NS equations. The general solution of the system can be written as

$$u_n = k_{n1} u_{n1} + k_{n2} u_{n2} + k_{n3} u_{n3}, \tag{29}$$

$$v_n = k_{n1} v_{n1} + k_{n2} v_{n2}, \tag{30}$$

$$p_n = k_{n1} p_{n1} + k_{n2} p_{n2}, \tag{31}$$

where k_{ni} are the three integration constants at any given order.

III. TERMS OF THE EXPANSIONS

The inner and outer expansion terms, which are solutions of systems (19) and (28), are here listed up to the fourth order. In the expansion, the integration constants, $\mathcal{C}_{in} = A, C_0, \ldots, C_4, \ldots, K_0, K_4, \ldots$ for the inner region and $\mathcal{C}_{on} = k_{n1}, k_{n2}, k_{n3}$ for the outer region, are determined by the boundary and matching conditions (5)–(8). It is found that the matching, at any given order, leads to a considerable simplification of the higher order system of equations. The sequence of the general and simplified systems of equations is given in Ref. 27. One should note that in the general system (28), the nonlinear and viscous terms are always confined to the inhomogeneous terms U_n, V_n, P_n, but actually enter into the system, at the second (r^{-1}) and third $(r^{-3/2})$ orders, respectively, whilst in the matched outer system, see Ref. 27, due to the peculiar simplification brought about by the matching, the nonlinear and viscous terms appear jointly only at the fifth order $(r^{-5/2})$. However, the nonlinear and the lateral diffusive effects are dominant in the inner layer from the first order of accuracy $(x^{-1/2})$ and are accompanied by the effects of the streamwise diffusion and pressure variation from the third and fourth order outwards, respectively. The overall picture of the field is that of a nonlinear convection and diffusion of vorticity in the inner layer transferred to the outer flow at a first step by a linear transport, which is active from the third to the fourth order, and at a second step by the nonlinearity and diffusion processes activated from the fifth order onward.

A. Order 0

Inner terms (order x^0):

$$\phi_0 = C_0, \tag{32}$$

$$\chi_0 = 0, \tag{33}$$

$$\pi_0 = K_0. \tag{34}$$

Outer terms (order r^0):

$$u_0 = k_{03}, \tag{35}$$

$$v_0 = k_{02}, \tag{36}$$

$$p_0 = k_{01}. \tag{37}$$

Both the inner system and the outer system have general solutions of the kind $u=$const, $v=$const, $p=$const. At this order, the matching conditions are trivially satisfied, and the correct solutions are directly determined by the boundary conditions at infinity: thus, $C_0 = 1, K_0 = k_{01} = \pi_\infty$, where $\pi_\infty = p_\infty/(\rho_\infty U^2)$, $k_{02} = 0, k_{03} = 1$.

B. First order

Inner terms (order $x^{-1/2}$):

$$\phi_1 = -A C_1 \mathrm{e}^{-(R/4)\,\eta^2}, \tag{38}$$

$$\chi_1 = 0, \tag{39}$$

$$\pi_1 = 0. \tag{40}$$

These terms give the well-known asymptotic Gaussian solution.[1,24,28] The function coefficients ϕ_1 and χ_1 tend to zero exponentially as $\eta \to \infty$. Considering $C_1 = -1$, the factorization constant A is given by the boundary condition (5) and the u distribution at $x = x_*$. Since the momentum defect in the wake does not depend on x this is equivalent to obtaining the value of A from the bluff body drag coefficient c_D, which results in $A(R) = \frac{1}{4}(R/\pi)^{1/2} c_D(R)$. Coefficients χ_1 and π_1 vanish identically, as can be seen from Eqs. (19) and (23).

Outer terms (order $r^{-1/2}$):

$$u_1 = i k_{11} \frac{s_+^{3/2}(1+is)^{1/2}}{3\,s\,(s+i)} - i k_{12} \frac{s_+^{5/2}(1+is)^{1/2}}{3\,s\,(s-i)^2} + k_{13} \frac{s_+}{s}, \tag{41}$$

$$v_1 = e^{(i/2)\,\arctan(s)} \left(k_{11} + k_{12} \frac{s_+}{s-i} \right), \tag{42}$$

$$p_1 = i e^{(i/2)\,\arctan(s)} \left(-k_{11} + k_{12} \frac{s+i}{s_+} \right). \tag{43}$$

Here the outer pressure in the inner limit $y \to 0$ gives $(\partial p/\partial x)_o \sim \frac{1}{2}(i k_{11} + k_{12}) + O(y)$, which immediately leads to $k_{12} = -i k_{11}$. At this order, it can be seen that the inner vorticity vanishes exponentially, but a check on the outer vorticity behavior in the inner limit $y \to 0$ shows that $\omega_o \sim (2k_{11}/3 + k_{13})\, x^{1/2} y^{-2} - (5k_{11}/12 + k_{13}/4)\, x^{-3/2} + O(y^2)$, therefore one sets $k_{13} = -(2/3) k_{11}$, and obtains $\omega_o \sim -(k_{11}/4)\, x^{-3/2} + O(y^2)$. This leads to $k_{11} = 0$, which gives the correct matching with the relevant inner term. Thus, the only acceptable physical solution is defined by $k_{11} = k_{12} = k_{13} = 0$. The entrainment matching condition on v is also trivially satisfied and yields

$$u_1 = 0, \quad v_1 = 0, \quad p_1 = 0. \tag{44}$$

C. Second order

Inner terms (order x^{-1}):

$$\phi_2 = A^2 e^{-(R/4)\eta^2}\left[C_{21}\ F_1\left(-\frac{1}{2},\frac{1}{2};\frac{R}{4}\eta^2\right) + e^{-(R/4)\eta^2} + \frac{1}{2}\sqrt{\pi R}\ \eta\,\mathrm{erf}\left(\frac{\sqrt{R}}{2}\eta\right)\right], \quad (45)$$

$$\chi_2 = -\frac{A}{2}\eta e^{-(R/4)\eta^2}, \quad (46)$$

$$\pi_2 = 0. \quad (47)$$

As $\eta\to\infty$, it can be shown that ϕ_2 has an exponential decay when $C_2=0$, otherwise $\phi_2\sim\eta^{-2}$; the behavior of χ_2 is $\chi_2\to 0$ exponentially.

Outer terms (order r^{-1}).

The relevant system has the general solution

$$u_2 = k_{21}s_- + k_{22}\frac{s_-}{s} + k_{23}\frac{s_+}{s}, \quad (48)$$

$$v_2 = k_{21}ss_- + k_{22}s_-, \quad (49)$$

$$p_2 = k_{22}ss_- - k_{21}s_-, \quad (50)$$

so that the inner behavior of the outer pressure is $(\partial p_x)_o \sim k_{21}x^{-2} + O(y)$, therefore one sets $k_{21}=0$. At this order, the inner vorticity vanishes exponentially if $C_2=0$, otherwise $\omega_i\sim$ const $C_2\,y^{-3}+O(y^{-\infty})$. The outer vorticity behavior in the inner limit is $\omega_o\sim(k_{22}+k_{23})y^{-2}+O(y^3)$, therefore the only correct matching is given by $C_2=0$ in the inner expansion and $k_{23}=-k_{22}$ in the outer. Therefore, in order to determine the value of k_{22}, one checks the behavior of the outer velocity v_o in the inner limit: $v_o = v_2/r \sim k_{22}/x$. Since, at the same order, $v_i\sim 0$, it follows that $k_{22}=0$, thus $k_{21}=k_{22}=k_{23}=0$ and

$$u_2 = 0, \quad v_2 = 0, \quad p_2 = 0. \quad (51)$$

D. Third order

Inner terms (order $x^{-3/2}$):

$$\phi_3 = A^3 e^{-(R/4)\eta^2}(2-R\eta^2)\left[\frac{1}{2}C_3 - RF_3(\eta)\right], \quad (52)$$

$$\chi_3 = \frac{A^2}{2}\Bigg\{ C_2\left[\int e^{-(R/4)\eta^2}{}_1\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{2};(R/4)\eta^2\right)d\eta + \frac{1}{2}\eta e^{-(R/4)\eta^2}{}_1\mathrm{F}_1\left(-\frac{1}{2},\frac{1}{2};(R/4)\eta^2\right)\right] + \frac{1}{2}\eta\, e^{-(R/2)\eta^2} + \sqrt{\frac{\pi}{2R}}\mathrm{erf}\left(\frac{\sqrt{R}}{2}\eta\right) - \left(\sqrt{\frac{\pi}{R}} - \frac{\sqrt{\pi R}}{4}\eta^2\right)e^{-(R/4)\eta^2}\mathrm{erf}\left(\frac{\sqrt{R}}{2}\eta\right)\Bigg\}, \quad (53)$$

$$\pi_3 = 0. \quad (54)$$

Here, we have $\phi_3\sim\eta^{-3}$ and $\chi_3\sim\chi_{3\,\infty}=A^2/2\sqrt{\pi/(2R)}$ as $\eta\to\infty$. In ϕ_3, the constant C_3 is determined by the boundary condition at $x=x_*$. In χ_3, the constant C_2 is zero as determined at the previous order.

Outer terms (order $r^{-3/2}$): the general solution of the relevant system is

$$u_3 = \frac{i}{3}k_{31}\,e^{(3\,i/2)\arctan(s)} + k_{33}\frac{s_+^{3/2}}{s^{3/2}} + \frac{1}{2}k_{32}\,s^{-3/2}s_+^{3/2} \times\Bigg\{ \frac{\sqrt{(1+i\,s)\,s}\left(\frac{3}{4}-\frac{i}{i+s}\right)}{2\,(i+s)} + \frac{(-1)^{1/4}}{16\sqrt{2}}\log \times\left[\frac{\left(\frac{i-1}{\sqrt{2}}+\sqrt{s}\right)\left(\frac{i-1}{\sqrt{2}}-(1-i)\sqrt{1+i\,s}+\sqrt{s}\right)}{\left(\frac{1-i}{\sqrt{2}}+\sqrt{s}\right)\left(\frac{1-i}{\sqrt{2}}+(1-i)\sqrt{1+i\,s}+\sqrt{s}\right)}\right]\Bigg\}, \quad (55)$$

$$v_3 = e^{(3/2)\,i\arctan(s)}\left[k_{31} + k_{32}s_+\frac{s+i}{3(s-i)^2}\right], \quad (56)$$

$$p_3 = e^{(3/2)\,i\arctan(s)}\left[-ik_{31} - k_{32}s_-\frac{(s+i)^2}{3(1+is)}\right]. \quad (57)$$

Here the inner behavior of the outer pressure is $(\partial p/\partial x)_o \sim \frac{1}{2}(3ik_{31}-k_{32})x^{-5/2}+O(y)$, which gives $k_{32}=3ik_{31}$. The outer limit of the inner vorticity is now $\omega_i\sim(\mathrm{const})\,x^{-5/2}+O(y^{-4})$, while the behavior of the outer vorticity in the inner limit is of the kind $\omega_o\sim(3k_{33}/2)y^{-5/2}+(\mathrm{const})\,k_{31}x^{-5/2}+O(y)$. The correct matching is obtained by setting $k_{33}=0$, which gives $\omega_o\sim(\mathrm{const})k_{31}x^{-5/2}+O(y)$. Finally, in order to determine the value of k_{31}, one can look at the behavior of the outer velocity v_o in the inner limit: $v_o\sim k_{31}x^{-3/2}$. The outer limit of the inner velocity v, at the same order, is $v_i\sim\chi_{3\,\infty}x^{-3/2}$, therefore one obtains $k_{31}=\pm\frac{1}{2}\chi_{3\,\infty}$, where the sign changes with y, since v is antisymmetric.

E. Fourth order

Inner terms (order x^{-2}):

$$\phi_4 = -A^4 e^{-(R/4)\eta^2}\left[C_4\,{}_1\mathrm{F}_1\left(-\frac{3}{2},\frac{1}{2};\frac{R}{4}\eta^2\right) + R^{3/2}\eta(R\eta^2-6)F_4(\eta)\right], \quad (58)$$

$$\chi_4 = \frac{\eta}{2}\phi_3 + \Phi_3, \quad (59)$$

$$\pi_4 = \int\left(\frac{1}{R}\chi_3'' + \frac{\eta}{2}\chi_3' + \frac{3}{2}\chi_3 + \phi_1\chi_2\right)d\eta + K_4. \quad (60)$$

Here, and in general for $n\geqslant 3$, we have $\phi_n\to 0$ with algebraic decay and $\chi_n\to$ const$\neq 0$ as $\eta\to\infty$. The behavior of π_4 is instead divergent: one in fact has $\pi_4\sim$ (const)$\eta+K_4$ (but the matching inner expansion of the outer term has the correct behavior, see the following). The constants C_4 and K_4 are determined by the boundary condition (5).

Outer terms (order r^{-2}):

$$u_4 = 2k_{41} s\, s_-^2 - \frac{1}{2} k_{42} \frac{3s^2+1}{s^2 s_+^2} + k_{43} \frac{s_+^2}{s^2}, \tag{61}$$

$$v_4 = k_{41} s_-^2 (s^2-1) - k_{42} s s_-^2, \tag{62}$$

$$p_4 = \frac{1}{2} k_{42} s_-^2 (1-s^2) - 2k_{41} s s_-^2. \tag{63}$$

The behavior of the pressure gradient in the inner limit is now $\partial_x p_o \sim -k_{42} x^{-3} + O(y)$, while the outer behavior of the inner pressure is $\partial_x p_i \sim$ (const)$y + K_4 x^{-3}$, therefore one sets $k_{42} = -K_4$. The comparison between the vorticities gives $\omega_o \sim (k_{42}+2k_{43}) y^{-3} + [(\mathrm{const}) k_{31} + O(x^{-1/2})] x^{-5/2} + O(y)$ in the inner limit and $\omega_i \sim [(\mathrm{const}) + O(x^{-1/2})] x^{-5/2} + O(y^{-4})$ in the outer limit. Thus, $k_{43} = -k_{42}/2$. Finally, the comparison between the lateral velocities gives $v_o \sim 2k_{31} x^{-3/2} - k_{41} x^{-2} + O(x^{-5/2})$ and $v_i \sim \chi_{3\infty} x^{-3/2} + \chi_{4\infty} x^{-2} + O(x^{-5/2})$, therefore one obtains $k_{41} = \pm \chi_{4\infty}$.

IV. TEST: THE CIRCULAR CYLINDER WAKE

The matched asymptotic expansion for the intermediate and far wake behind the circular cylinder can be determined by setting the inner and outer coefficients according to the boundary and matching conditions (5),(6)–(8) as discussed in Sec. III. In particular it is important to use the experimental field information, as already mentioned by Stewartson in 1957,[13] fixing the flow distributions at a given section of the intermediate region see Sec. II [(4) and (5)]. The inner and outer expansions are then used to form the composite expansion f_{cn}—which, by construction, is continuous and differentiable over the whole domain—acccording to the rule $f_{cn} = f_{in} + f_{on} - (f_{on})^{in}$, where $(f_{on})^{in}$ is the common part of f_{in} and f_{on}, that can be calculated as the inner expansion of the outer expansion, or vice versa.

The inner, outer, and composite distributions of the velocity components and pressure, at $R=34$ and at twenty diameters downstream to the circular cylinder, are shown in Figs. 2, 3, and 4. Note that, in Fig. 2, it has been necessary to use a very enlarged ordinate scale to present the matching between u_i and u_o, which would not be visible otherwise.

In a previous work,[18] the wake—the inner field of the present problem—was studied through the boundary layer model. This simpler model allows the general order term of the expansion to be analytically determined. In this case, the pressure field is a uniform field, this being a limiting feature of the model. On the other hand the entrainment is very efficiently accounted for by the outer limit of the v field which has nonzero values very close to those issued by the Navier–Stokes model. On the contrary, seeking the solution in terms of matched NS expansions allows the lateral variations of the outer v field as well as the longitudinal and lateral variations of both the inner and outer pressure field to be obtained. See Fig. 5 for a comparison of the two models at $x=20,40D$ and $R=34$. In the case of the matched asymptotic expansion, it is interesting to observe that the inner NS pressure field is just of the fourth order [i.e., $O(x^{-2})$] and diverges at its outer limit, which makes the matching process with an outer field absolutely necessary.

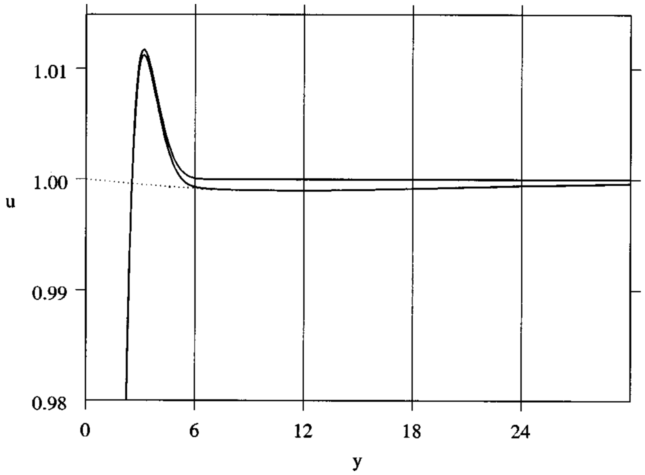

FIG. 2. $R=34$, $x/D=20$. Details of the matching between u_i (thin line) and u_o (dotted line) and of the composite u_c (thick line) expansion solution. Fourth order of accuracy.

Coming back to the matched solution for the longitudinal velocity component, it is possible to find, in Fig. 6, a comparison with: (1) the asymptotic matched solution by Chang,[15] (2) the experimental distribution by Kovasznay,[19] (3) the numerical distribution by Berrone.[20,21] Chang's solution, being the one that reaches the highest order of accuracy in both the inner and outer approximations, is here assumed as the reference for the ensemble of expansion solutions based on the rapid decay principle. It should be recalled that the latest one (Kida, 1984),[16] which was obtained under the same body of assumptions and which results in the same sequence of terms, is one order of accuracy lower than Chang's solution.

As can be seen in Fig. 6, the present matching turns out to be efficacious, contrasting positively with both the experimental and numerical profiles. Chang's results however do not fully agree with these distributions. This is due to the

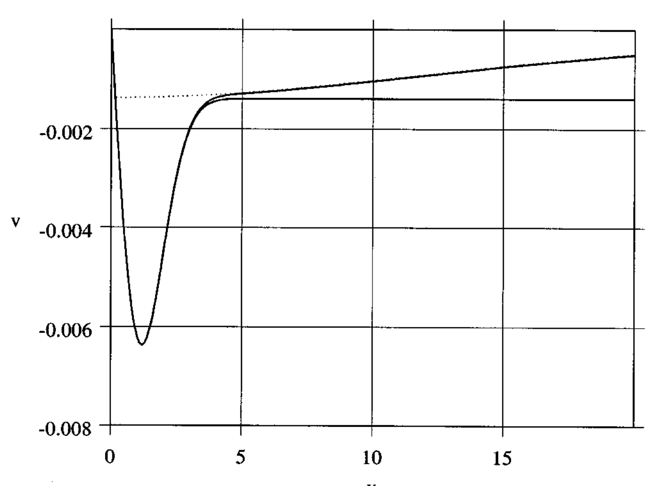

FIG. 3. $R=34$, $x/D=20$. Matching between v_i (thin line) and v_o (dotted line); composite v_c (thick line) present expansion solution. Fourth order of accuracy.

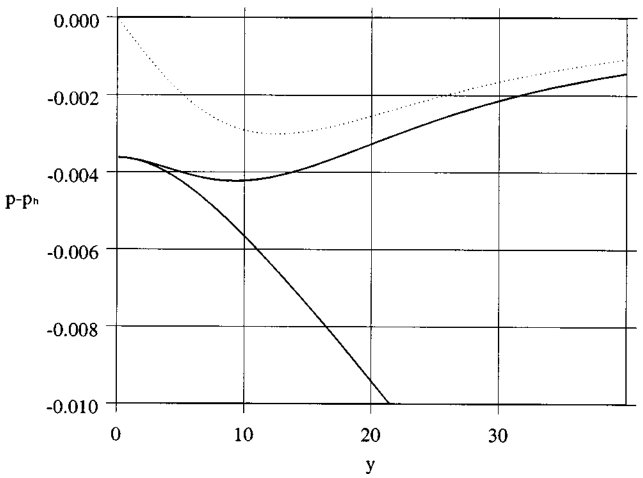

FIG. 4. $R=34$, $x/D=20$. Matching between $p_i - p_{\mathrm{inf}}$ (thin line) and $p_o - p_{\mathrm{inf}}$ (dotted line); composite p_c (thick line) present expansion solution. Fourth order of accuracy.

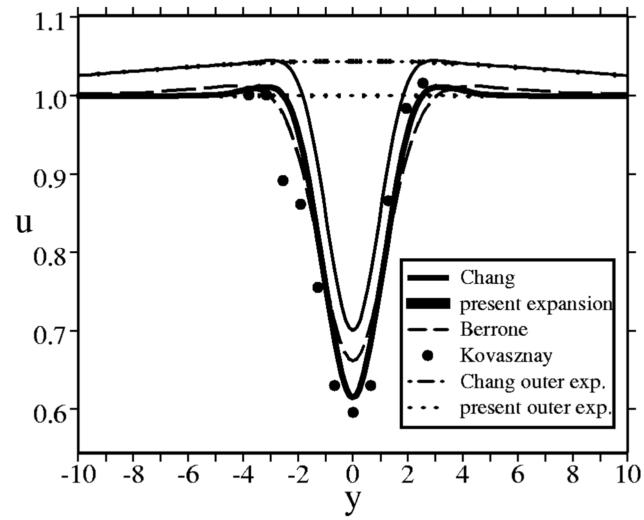

FIG. 6. $R=34$, $x/D=20$. Comparison of the present (outer and composite), Chang's (outer and composite), Kovasznay's experimental and Berrone's numerical longitudinal velocity distributions. The field information, condition (5), see Sec. II A, for the present solution and coefficients m, a_2, c_2 for Chang's solution (Chang, 1961), has been inserted using the tabulated data set in Kovasznay (1948), which is, even these days, one of the richest sources of experimental velocity profiles for the steady circular cylinder wake.

presence, in his inner expansion, of a term const/$2\pi x$ inside the group of terms at $O(\epsilon^{1/2})$. Here ϵ is the small artificial parameter used by Chang in his expansion, which may be eliminated from the problem by reformulating it in terms of the physical coordinates. Chang's term stems from the assumption that the field can accommodate an inner pressure which is independent of the lateral coordinate, which however varies at the leading orders along the x coordinate. According to Chang, the presence of this term should not be a problem because it is also common to the outer solution. In the matching, it is subtracted from the inner expansion, and only remains in the outer expansion, hidden inside terms like const/$2\pi z$, which, in the outer limit $z\to\infty$, $z=x+iy$, tends to zero. However, at intermediate values of y and for fixed x, this type of term is responsible for an anomalous rise in the composite expansion, due to the central plateau that is present in the outer expansion. This is visible in the plots of Fig. 6, where, inside the interval $(-10<y<10)$, the present expansion is compared to Chang's third-order approximation $[O(\epsilon^{3/2})]$, which is equivalent to $O(x^{-3/2})$ and to $O(r^{-3/2})$ for the present inner and outer expansions, respectively. In Fig. 6 Chang's distributions clearly show the presence of the term $\epsilon(\mathrm{const}/2\pi x)$, which biases the outer solution at finite values of x to values greater than 1 and forces the composite expansion to assume inaccurate values mostly in the region around $y/D\approx 2$ and outwards (at $y/D=20$ the longitudinal velocity is still appreciably different from U).

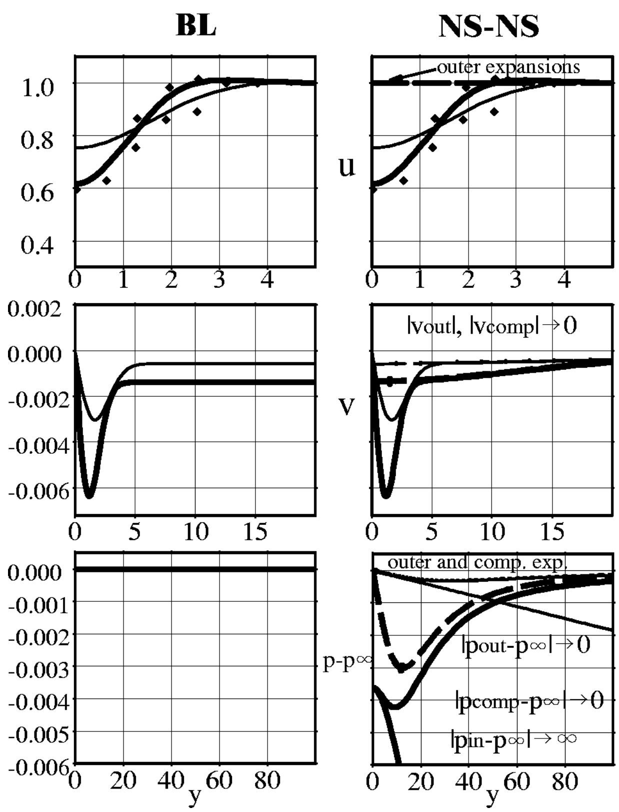

FIG. 5. $R=34$, $x/D=20,40$. Comparison between the boundary layer expansion (Ref. 18) and the present matched solution. Thick lines: $x=20$, thin lines: $x=40$. Full lines: BL profiles; NS–NS matched solution: inner pressure and composite profiles. Broken lines: outer profiles. The diamonds are data by Kovasznay (1948) (Ref. 19), which are, even these days, the only source of experimental u velocity profiles for the intermediate region of the steady circular cylinder wake. The Kovasznay data do not include information on the lateral velocity and the pressure.

Other remarks could be made. Chang's matching is directly conducted on the velocity and pressure and sometimes also on the stream function. The terms of the outer expansion are obtained from those of the inner expansion through analytical extrapolation (the principle of eliminability of the artificial parameter ϵ coupled to a switchback procedure, which, however, has not been generalized for terms involving log ϵ) but these terms were not checked to be solutions at the various orders of the outer flow equations (which were not presented in the aforementioned paper). In spite of the fact that the inner expansion is forced to assume a lateral exponential decay, the higher order terms (e.g., the order

$\epsilon^{3/2}$) of the outer expansion are rotational. Presumably Chang just expected irrotationality for the sum of the terms of the whole outer asymptotic expansion. However, leaving aside the problem of proof, which probably cannot be found, it should be recalled that an unavoidable truncation affects any actual approximation.

V. CONCLUSIONS

A Navier–Stokes inverse coordinate expansion solution is here presented for steady two-dimensional wakes of bluff bodies. It is a matching between inner and outer Navier–Stokes asymptotic expansions calculated behind the body in the domain $d<x<\infty$, $-\infty<y<\infty$, where $d=d(R)$ is the suitable left limit of the intermediate asymptotics of the field. In this region the wake can be considered a thin layer. The approximated solution was sought in the range of Reynolds numbers close to the critical value for the onset of the first instability and for the intermediate part of the flow where the nonparallelism of the streamlines is still appreciable and, as a consequence, the convection terms in the equation of motion should not yet be linearized, as instead is the case in the Oseen representation of the far wake. The expansion solution also holds for the far wake which asymptotically coincides with the Gaussian representation, which is an Oseen solution. The new solution presented here exhibits two properties: (i) analytical simplicity, which makes a simple but detailed basic flow available for the study of the instability of the nonparallel portion of bluff body wakes, and (ii) good agreement with experimental data. Nevertheless, the most important result concerning this solution is, in the authors' opinion, the fact that it has been obtained by relaxing the exponential decay principle for the inner layer, whose addition to the governing equations, on one hand, restricts their generality, while on the other makes the introduction of logarithmic terms in the expansion necessary. The present approach, however, did not prevent the matching, which was based on criteria that involve the joining of the longitudinal pressure gradient, vorticity, and entrainment velocity, from spontaneously showing the properties of rapid decay and irrotationality at the first and second orders of accuracy for the inner and the outer flows, respectively. At the higher orders this approach leads to a fast algebraic decay of the inner layer and to an outer flow, which, up to the order r^{-2}, linearly convects momentum and, from the order $r^{-5/2}$, nonlinearly convects and diffuses it.

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