# Self-similarity of the turbulence mixing with a constant macroscale gradient

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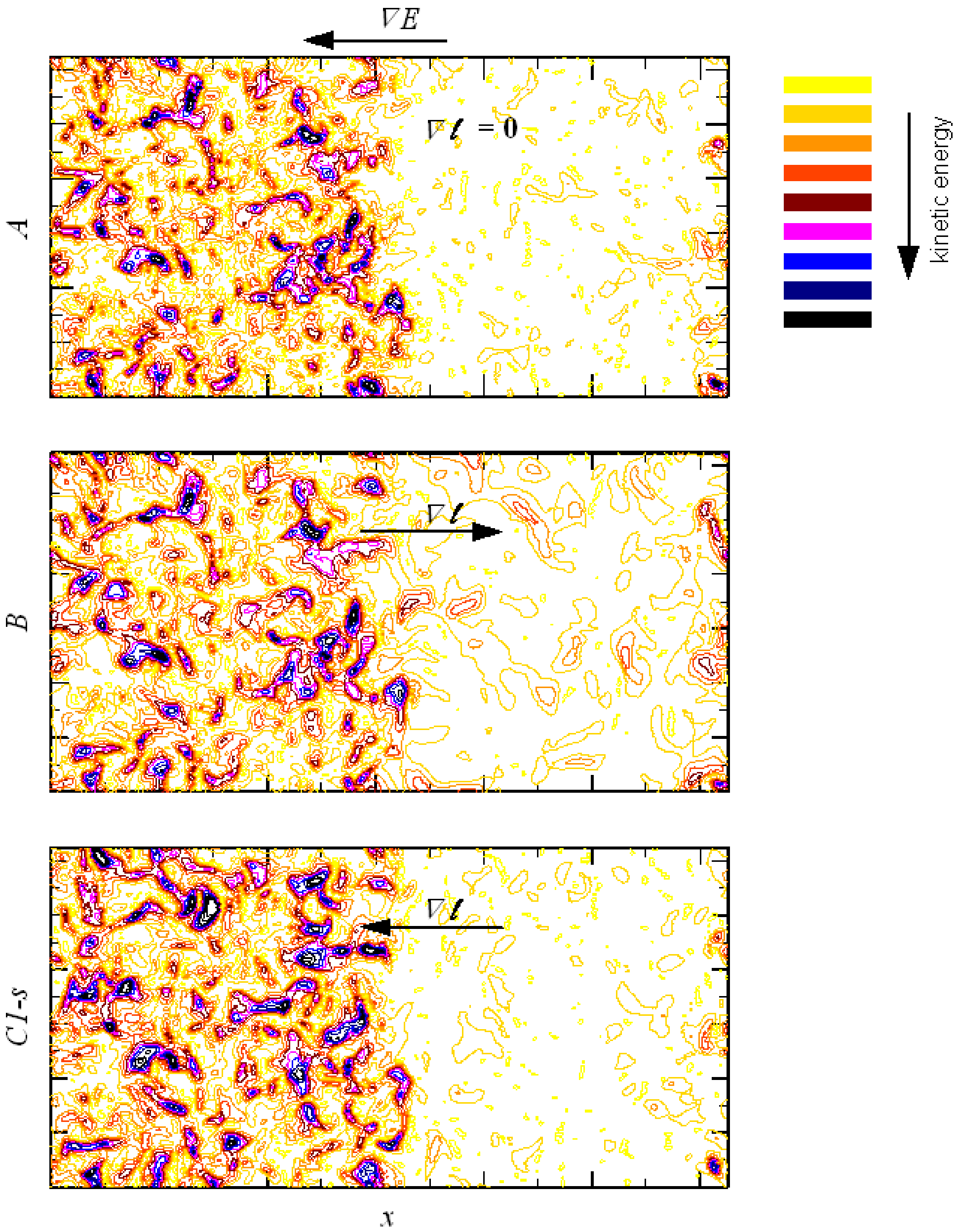
# Shearless turbulence mixing.



## Part 1: numerical experiments

Numerical simulations (DNS and LES) have been carried out with

- Energy ratio $1 < \mathcal{E} \leq 58.3$
- Scale ratio $0.6 \leq \mathcal{L} \leq 2.1$
- Reynolds number: $Re_\lambda \approx 45$ (DNS, LES) and $Re_\lambda \approx 450$ (LES)
- Numerical method: Fourier-Galerkin pseudospectral on a $2\pi \times 2\pi \times 4\pi$ parallelepiped
  Resolution: DNS $= 128^2 \times 256$, LES $= 32^2 \times 64$
- Initial conditions: two homogeneous turbunent fields ...

## flow parameters

- Homogeneous decaying turbulent fields

$$E = A(t + t_0)^{-n} \tag{1}$$

- Similar decay rates $n_1 \approx n_2$: energy and scale ratio remain constant

$$\frac{\mathcal{L}(t)}{\mathcal{L}(0)} = \left(1 + \frac{t}{t_{01}}\right)^{1 - \dfrac{n_1}{2}} \left(1 + \frac{t}{t_{02}}\right)^{-1 + \dfrac{n_2}{2}} \tag{2}$$

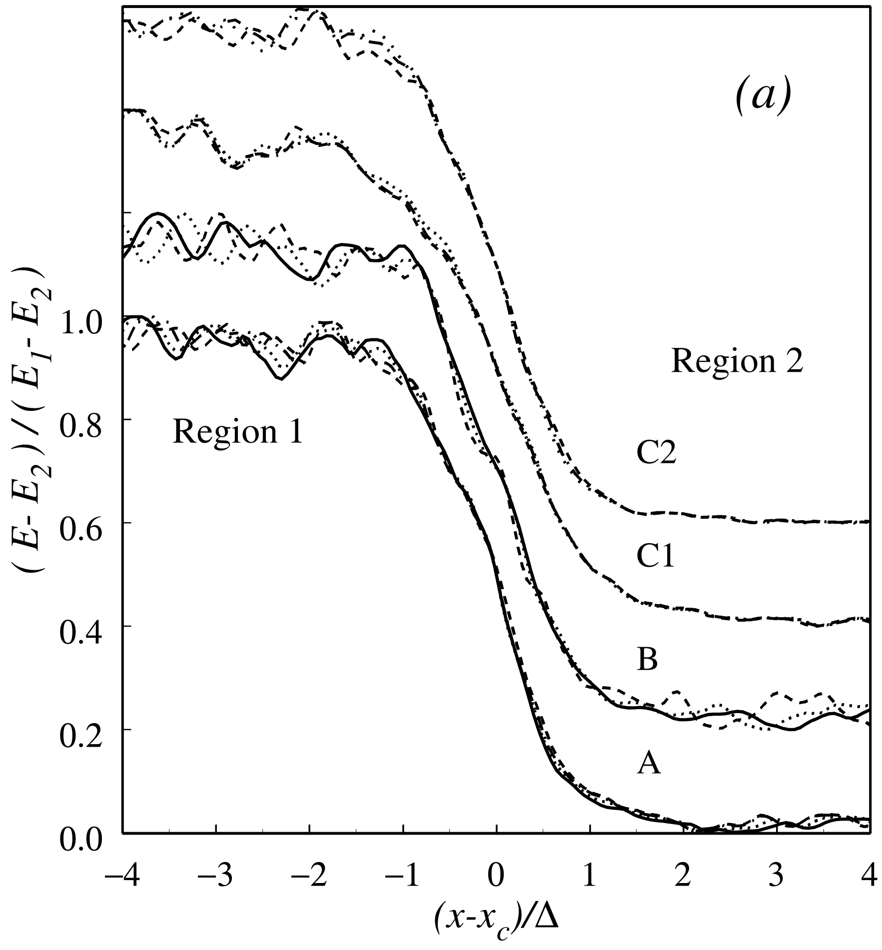
$$\frac{\mathcal{E}(t)}{\mathcal{E}(0)} = \left(1 + \frac{t}{t_{02}}\right)^{n_2} \left(1 + \frac{t}{t_{01}}\right)^{-n_1} \tag{3}$$

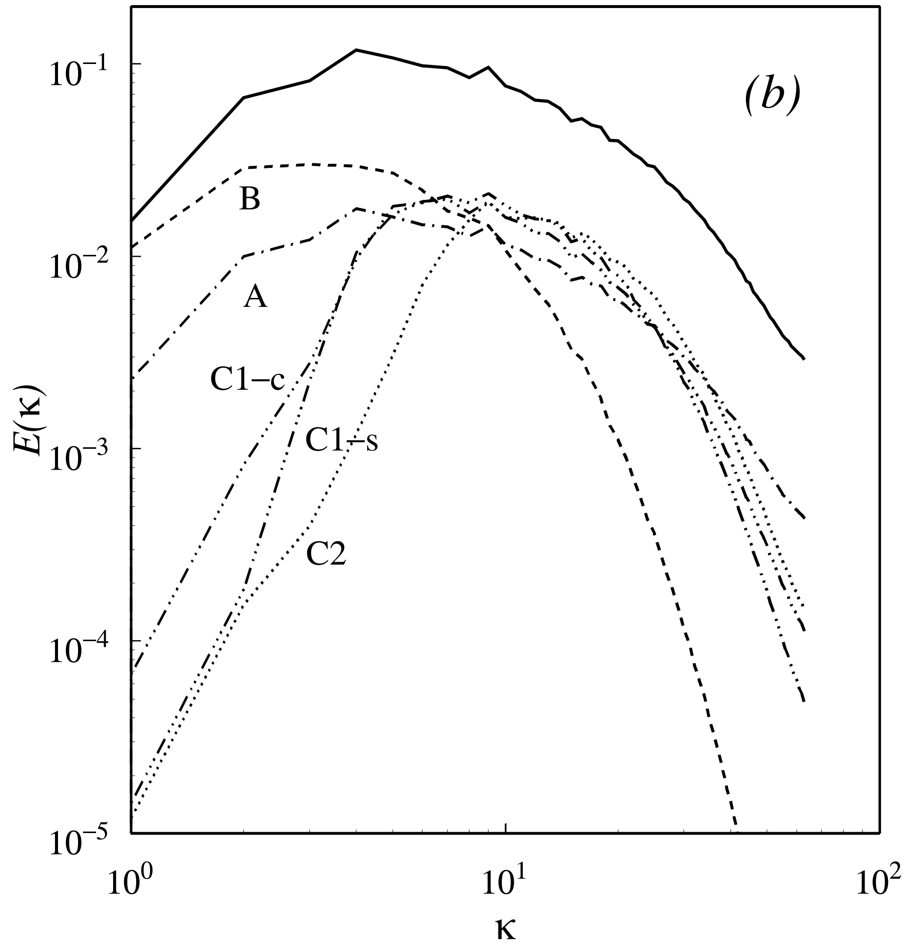
- $Re_\lambda \approx 45$

| | | $\mathcal{E} = \frac{E_1}{E_2}$ | $\nabla_*(E/E_1)$ | $\mathcal{L} = \frac{\ell_1}{\ell_2}$ | $\nabla_*(\ell/\ell_1)$ | $Re_{\lambda_1}$ | $n_1$ | $n_2$ | $\frac{x_s - x_c}{\Delta}$ | $\frac{x_k - x_c}{\Delta}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| DNS: | A | 6.7 | 0.425 | 1.0 (1.6) | 0.0 | 45.4 | 1.22 | 1.16 | 0.42 | 0.62 |
| | B | 6.6 | 0.424 | 0.6 (0.5) | 0.33 | 45.4 | 1.22 | 1.32 | 0.36 | 0.45 |
| | C1-s | 6.6 | 0.424 | 1.5 (1.7) | -0.17 | 45.4 | 1.22 | 1.39 | 0.63 | 0.84 |
| | C1-c | 6.5 | 0.423 | 1.5 (1.8) | -0.17 | 45.4 | 1.22 | 1.37 | 0.63 | 0.84 |
| | C2 | 6.5 | 0.423 | 2.1 (2.7) | -0.26 | 45.4 | 1.22 | 1.56 | 0.79 | 0.98 |
| LES: | a1 | 1.43 | 0.149 | 1.0 | 0.0 | 45 | 1.25 | 1.16 | 0.17 | 0.30 |
| | a2 | 6.7 | 0.421 | 1.0 | 0.0 | 45 | 1.25 | 1.15 | 0.54 | 0.75 |
| | a3 | 12.1 | 0.454 | 1.0 | 0.0 | 45 | 1.23 | 1.15 | 0.65 | 0.86 |
| | | 12.1 | 0.454 | 1.0 | 0.0 | 450 | 1.20 | 1.14 | 0.72 | 0.93 |
| | a4 | 24.0 | 0.474 | 1.0 | 0.0 | 45 | 1.22 | 1.13 | 0.82 | 0.99 |
| | | 24.0 | 0.474 | 1.0 | 0.0 | 450 | 1.20 | 1.13 | 0.86 | 1.06 |
| | a5 | 58.1 | 0.485 | 1.0 | 0.0 | 45 | 1.22 | 1.13 | 1.07 | 1.23 |
| | | 58.1 | 0.485 | 1.0 | 0.0 | 450 | 1.20 | 1.13 | 1.19 | 1.28 |
| | b1 | 24.0 | 0.474 | 0.53 | 0.0 | 45 | 1.24 | 1.35 | 0.72 | 0.86 |
| | b2 | 58.0 | 0.485 | 0.38 | 0.0 | 45 | 1.20 | 1.11 | 0.80 | 0.91 |
| V&W - bars | | 6.2 | | (2.4) | | 78.1 | 1.22 | 1.39 | 0.30 | 0.30 |
| V&W - plate | | 6.3 | | (2.2) | | 44.5 | 1.43 | 1.25 | 0.63 | 0.81 |
| Briggs *et al.* | | 7.5 | | 1.0(1.7) | | 40.3 | 1.55 | 1.35 | 0.38 | 0.51 |
| Knaepen *et al.* | | 6.27 | | (2.2) | | 69.0 | 1.30 | 1.10 | 0.77 | 1.06 |

Tabella 1: Flow parameters. E=kinetic energy, $\ell$=integral scale (eq.2.1), $Re_\lambda$=Taylor's microscale Reynolds number, $\nabla_* = \partial/\partial(x/\Delta)$=gradient normalized with the mixing layer thickness $\Delta$, $n$=exponent of the energy decay, $x_s$ and $x_k$ are the positions of the maxima of skewness and kurtosis. Index 1 refers to the high energy region, index 2 to the low energy region. In the fourth column, the data in parenthesis refer to scales computed through eq. 2.2.

# Energy similarity profiles

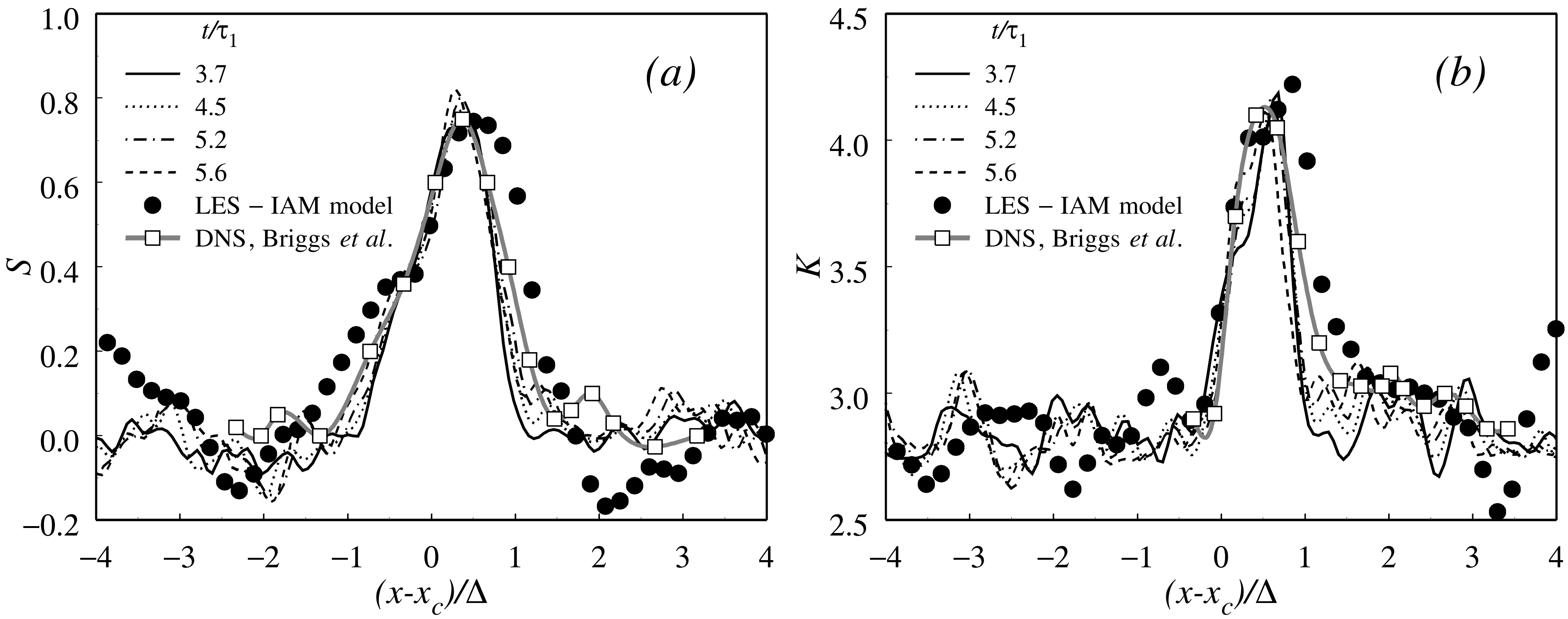


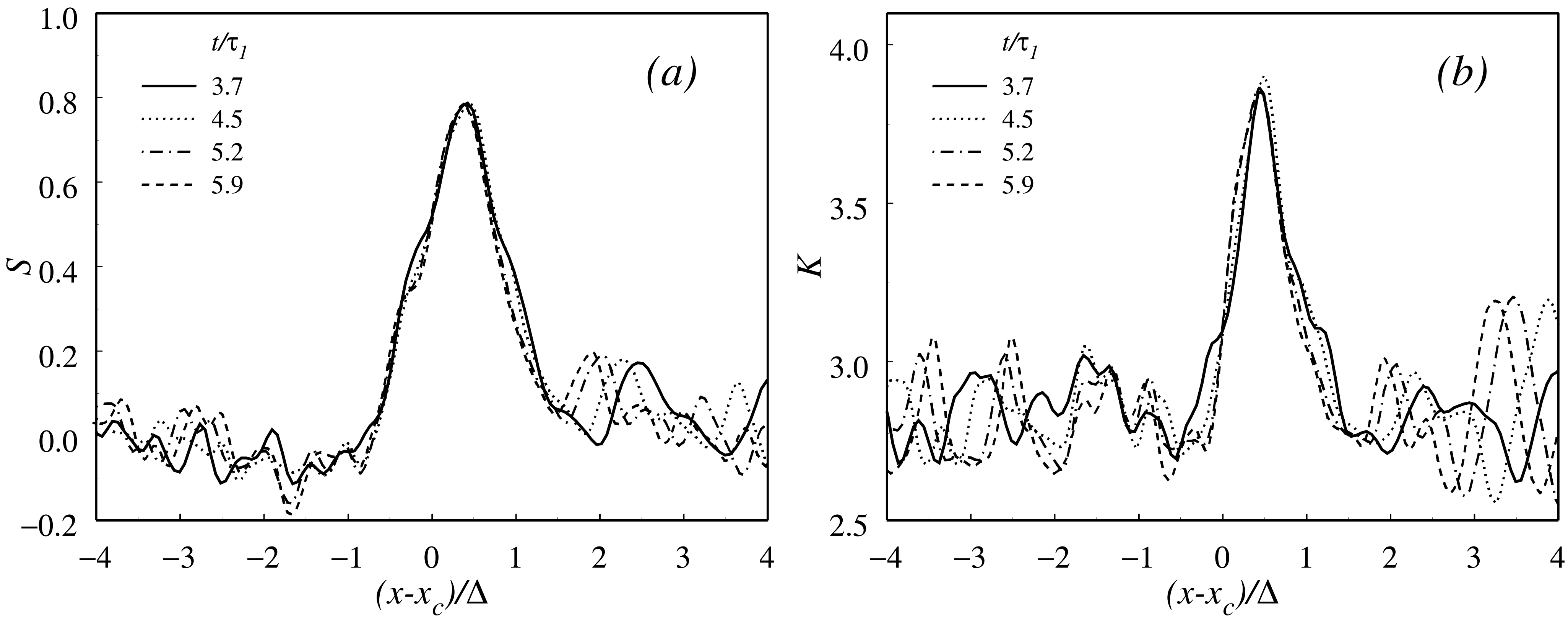
# Higher order moments: skewness and kurtosis profiles

$$S = \frac{\overline{u^3}}{\overline{u^2}^{\frac{3}{2}}} \quad K = \frac{\overline{u^4}}{\overline{u^2}^{2}}$$

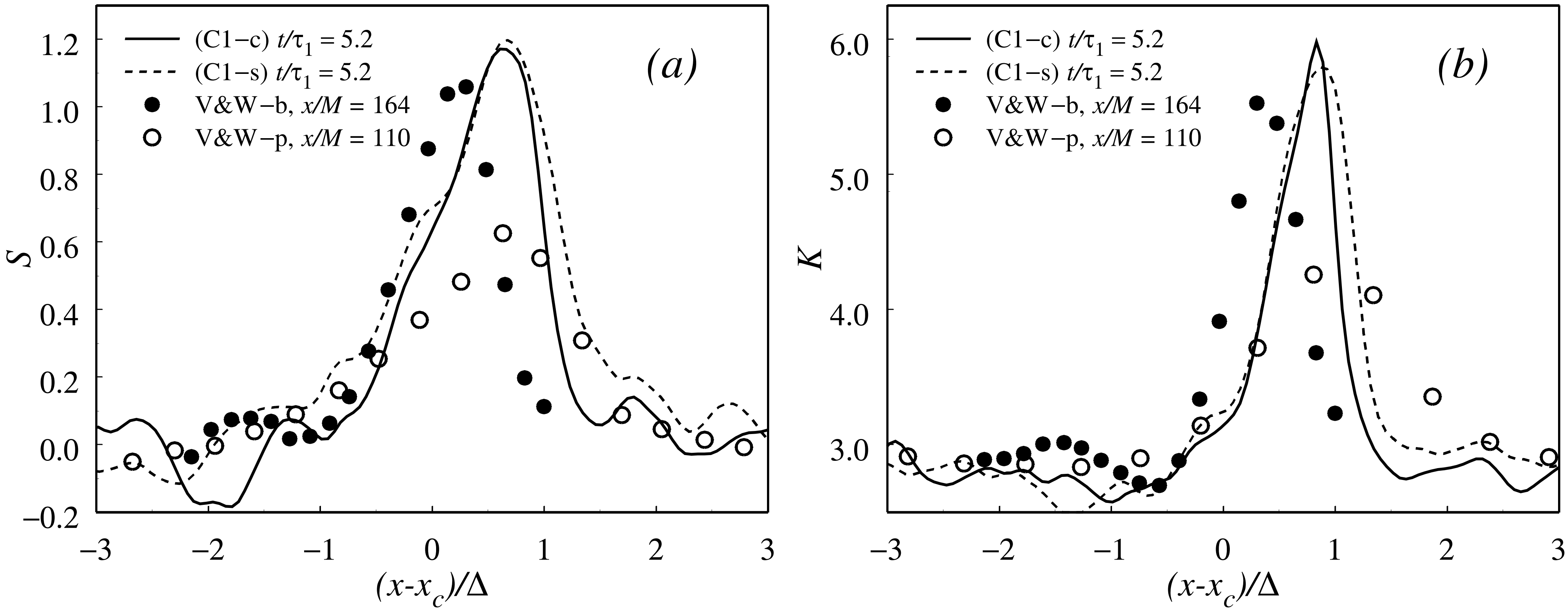
**Case A:** $\mathcal{E} = 6.7$, $\mathcal{L} = 1$, the two fields have same integral scale.



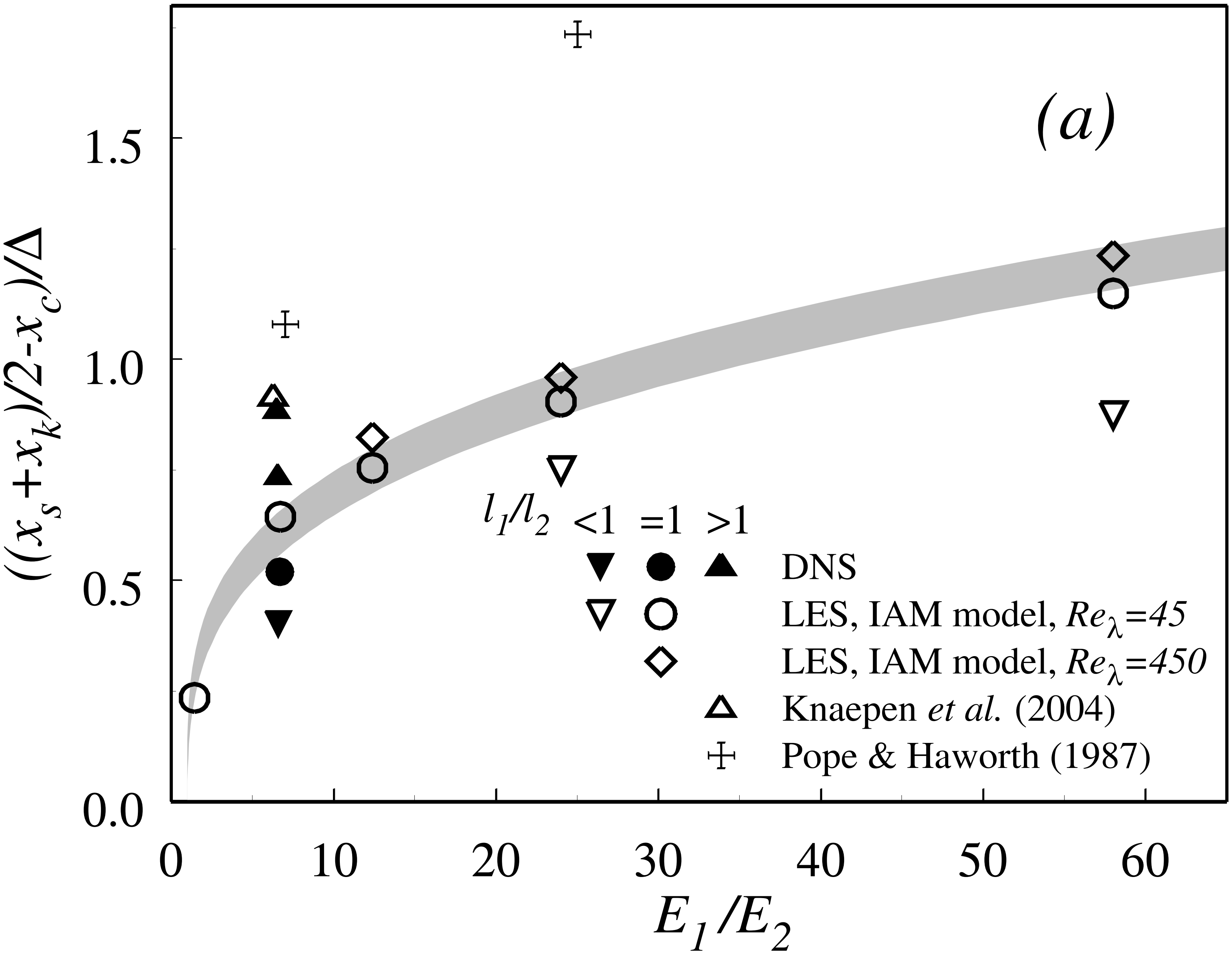
**Case B:** $\mathcal{E} = 6.6$, $\mathcal{L} = 0.6$, the gradients of energy and scales are opposite: larger scale turbulence has less energy

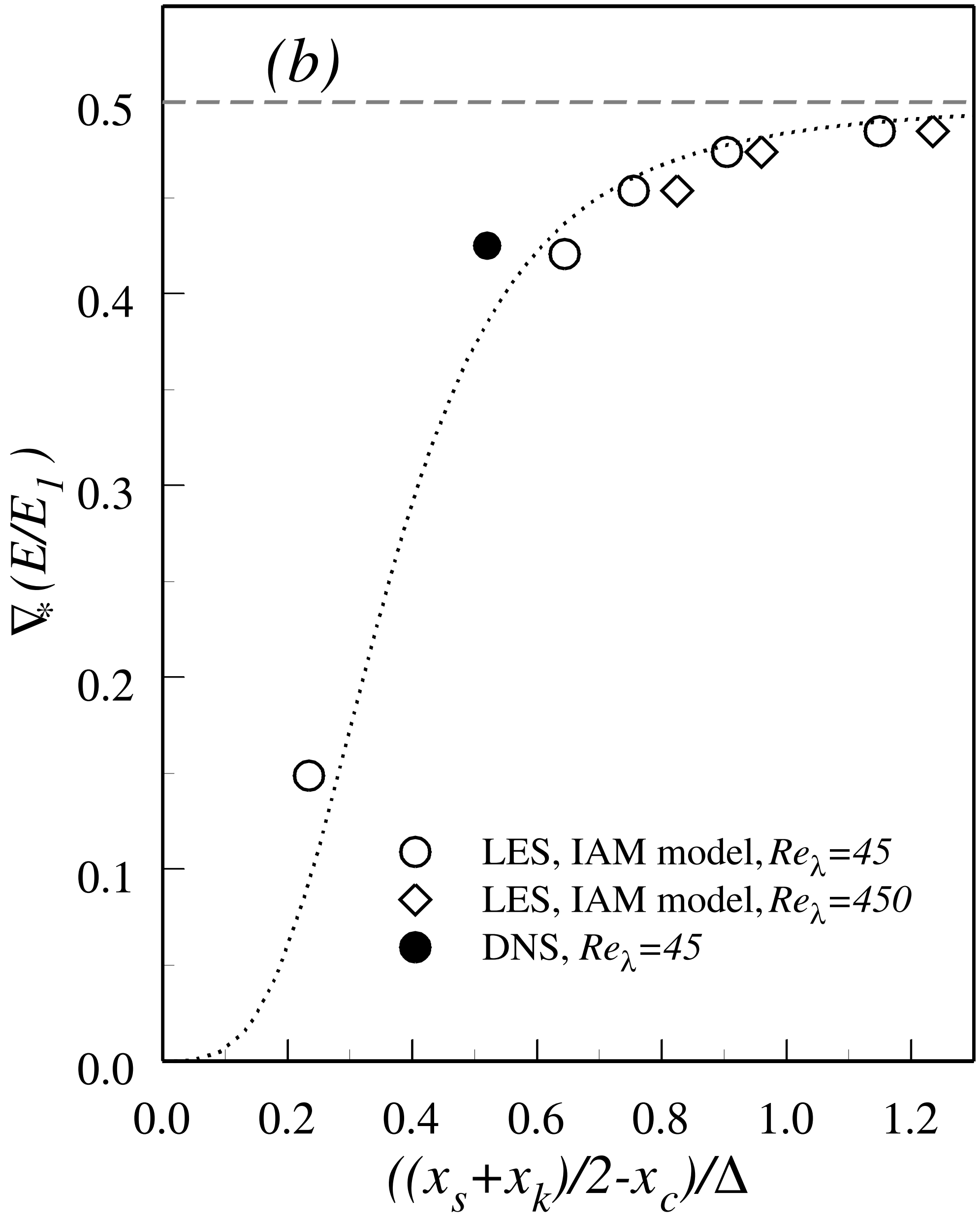


**Case C:** $\mathcal{E} = 6.5$, $\mathcal{L} = 1.5$: the gradients of energy and scales have the same sign: larger scale turbulence has more energy



## Penetration - position of maximum of skewness/kurtosis






Penetration with $\mathcal{L} = 1$

Scaling law (energy ratio):

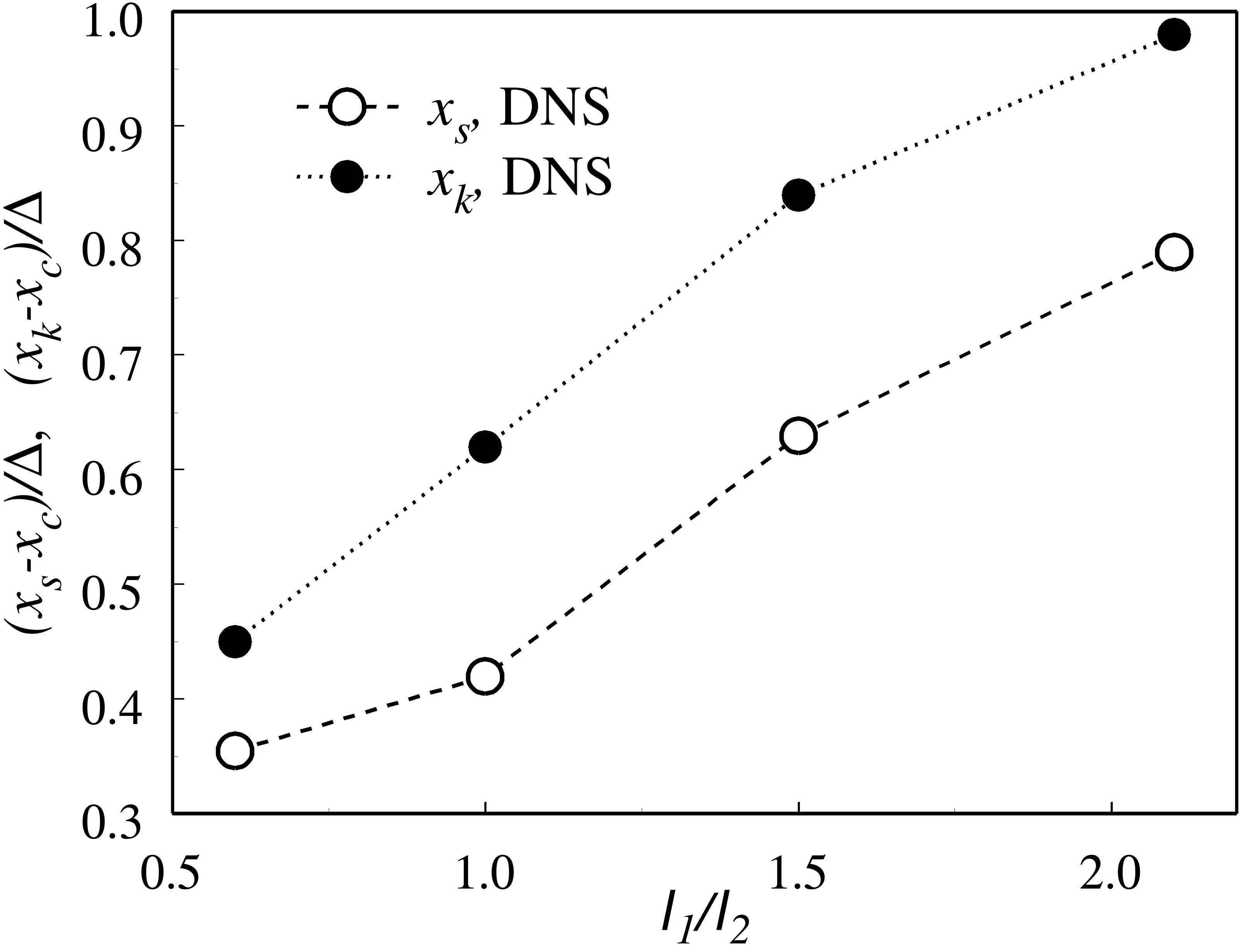
$$\frac{\eta_s + \eta_k}{2} \sim a\left(\frac{E_1}{E_2} - 1\right)^b$$

$$a \simeq 0.36, \quad b \simeq 0.298$$

Scaling law (energy gradient):

$$\nabla_*(E/E_1) \simeq (1 - \mathcal{E}^{-1})/2$$

$$\frac{\eta_s + \eta_k}{2} \sim a\left(\frac{2\nabla_*(E/E_1)}{1 - 2\nabla_*(E/E_1)}\right)^b$$



x_s, DNS
x_k, DNS
(x_s-x_c)/Δ, (x_k-x_c)/Δ
l_1/l_2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.5
1.0
1.5
2.0

## ???

- bla bla
- bla

## <u>Part 2:</u> similarity solutions

To carry out the similarity analysis, we considered the second order moment equations for single-point velocity correlations

$$\partial_t \overline{u^2} + \partial_x \overline{u^3} = -2\rho^{-1}\partial_x \overline{pu} + 2\rho^{-1}\overline{p\partial_x u} - 2\varepsilon_u + \nu\partial_x^2 \overline{u^2} \quad (4)$$

$$\partial_t \overline{v_1^2} + \partial_x \overline{v_1^2 u} = 2\rho^{-1}\overline{p\partial_{y_1} v_1} - 2\varepsilon_{v_1} + \nu\partial_x^2 \overline{v_1^2} \quad (5)$$

$$\partial_t \overline{v_2^2} + \partial_x \overline{v_2^2 u} = 2\rho^{-1}\overline{p\partial_{y_2} v_2} - 2\varepsilon_{v_2} + \nu\partial_x^2 \overline{v_2^2} \quad (6)$$

where:
$u$ is the velocity fluctuation in the inhomogeneous direction $x$,
$v_1, v_2$ are the velocity fluctuations in the plane $(y_1, y_2)$ normal to $x$,
$\varepsilon$ is the dissipation.

**boundary conditions:**

outside the mixing, turbulence is homogeneous and isotropic:

- For $x \to -\infty$ (high-energy turbulence):

$$\overline{u^2} = \overline{v_1^2} = \overline{v_2^2} = \frac{2}{3} E_1(t)$$

$$\overline{pu} = \overline{u^3} = \overline{v_1^2 u} = \overline{v_2^2 u} = 0$$

- For $x \to +\infty$ (low-energy turbulence):

$$\overline{u^2} = \overline{v_1^2} = \overline{v_2^2} = \frac{2}{3} E_2(t)$$

$$\overline{pu} = \overline{u^3} = \overline{v_1^2 u} = \overline{v_2^2 u} = 0$$

## Hypothesis and semplifications

- The two homogenous turbulences decay in the same way, thus

$$E_1(t) = A_1(t+t_0)^{-n_1}, \quad E_2(t) = A_2(t+t_0)^{-n_2}$$

the exponents $n_1$, $n_2$ are close each other (see experiments, Tordella & Iovieno, 2004). Here, we suppose $n_1 = n_2 = n = 1$, a value which corresponds to $R_\lambda \gg 1$ (Batchelor & Townsend, 1948).

- In the absence of energy production, the pressure-velocity correlation has been shown to be approximately proportional to the convective fluctuation transport (Yoshizawa, 1982, 2002)

$$-\rho^{-1}\overline{pu} = a\,\frac{\overline{u^3} + 2\overline{v_1^2 u}}{2}, \quad a \approx 0.10,$$

- Single-point second order moments are almost isotropic through the mixing:

$$\overline{u^2} \simeq \overline{v_i^2}$$

hypothesys (2) and (3) implies that pressure-velocity correlations can be expressed in terms of third order velocity correlations:

$$\overline{u^2} \simeq \overline{v_i^2}, \;\Rightarrow\; \overline{u^3} - \overline{v_1^2 u} \simeq 2\rho^{-1}\overline{p\partial_x u}$$

then

$$-\rho^{-1}\overline{pu} = \alpha\overline{u^3}, \quad \alpha = \frac{3a}{1+2a} \approx 0.25. \tag{7}$$

## Similarity hypothesis

The moment distributions in the above problem are determined by

- the coordinates $x$, $t$
- the energy $E_1(t)$, $E_2(t)$ of the two mixing turbulences.
- the scales $\ell_1(t)$, $\ell_2(t)$ of the two mixing turbulences.

Thus, through dimensional analysis,

$$\overline{u^2} = E_1^{\frac{1}{2}} \varphi_2(\eta, R_{\ell_1}, \vartheta_1, \mathcal{E}, \mathcal{L}) \tag{8}$$

$$\overline{u^3} = E_1 \varphi_{uuu}(\eta, R_{\ell_1}, \vartheta_1, \mathcal{E}, \mathcal{L}) \tag{9}$$

$$\varepsilon_u = E_1^{\frac{3}{2}} \ell_1^{-1} \varphi_{\varepsilon_u}(\eta, R_{\ell_1}, \vartheta_1, \mathcal{E}, \mathcal{L}), \tag{10}$$

where:

$\eta = x/\Delta(t)$, $\Delta(t)$ is the mixing layer thickness, $R_{\ell_1} = E_1^{\frac{1}{2}}(t)\ell_1(t)/\nu$,
$\vartheta_1 = tE_1^{\frac{1}{2}}(t)/\ell_1(t)$ $\mathcal{E} = E_1(t)/E_2(t)$, $\mathcal{L} = \ell_1(t)/\ell_2(t)$

The high Reynolds number algebraic decay ($n = 1$), implies:

$$\mathcal{E} = \text{const} = \frac{E_1(0)}{E_2(0)} \quad (11)$$

$$\mathcal{L} = \text{const} = \frac{\ell_1(0)}{\ell_2(0)} \quad (12)$$

$$\vartheta_1 = \text{const} = \frac{n}{f(R_{\lambda_1})} \quad (13)$$

$$R_{\ell_1} \propto t^{1-n} = \text{const} \quad (14)$$

$\Rightarrow \eta$ is the only similarity variable function of $x, t$.

## ⇒ similarity conditions:

After having introduced the similarity relations into equation ??, all coefficient must be independent from $x, t$, so that

$$\Delta(t) \propto \ell_1(t)$$

## ⇒ similarity equation:

$$-\frac{1}{2}\eta\frac{\partial \varphi_{uu}}{\partial \eta} + \frac{1}{f(R_{\lambda_1})}(1-2\alpha)\frac{\partial \varphi_{uuu}}{\partial \eta} + \frac{\nu}{Af(R_{\lambda_1})^2}\frac{\partial^2 \varphi_{uu}}{\partial \eta^2} = \\ = \varphi_{uu} - \frac{2}{f(R_{\lambda_1})}\varphi_{\varepsilon_u} \quad (15)$$

$\Rightarrow$ *third-order moments* can be expressed a function of *second order moments*: the skewness is then

$$\varphi_{uuu} = \frac{1}{(1-2\alpha)} \left[ \frac{f}{2} \int_{-\infty}^{\eta} \eta \frac{\partial \varphi_{uu}}{\partial \eta} \mathrm{d}\eta + \frac{\nu}{A_1 f} \frac{\partial \varphi_{uu}}{\partial \eta} + \beta f \frac{\partial^3 \varphi_{uu}}{\partial \eta^3} \right] \quad (16)$$

$$S = \frac{\varphi_{uu}^{-\frac{3}{2}}}{(1-2\alpha)} \left[ \frac{f}{2} \int_{-\infty}^{\eta} \eta \frac{\partial \varphi_{uu}}{\partial \eta} \mathrm{d}\eta + \frac{\nu}{A_1 f} \frac{\partial \varphi_{uu}}{\partial \eta} + \beta f \frac{\partial^3 \varphi_{uu}}{\partial \eta^3} \right] \quad (17)$$

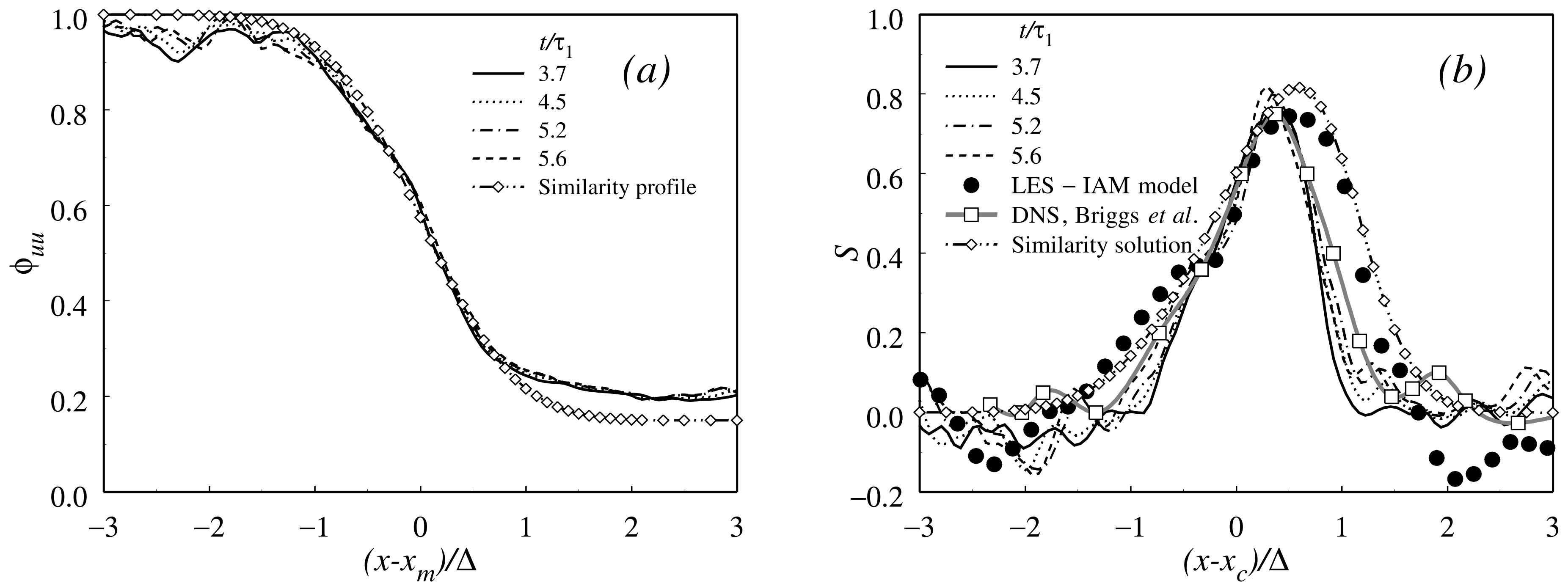
Numerical experiments suggest the following fit for second-order moments (see also Veeravalli & Wahrhaft, *JFM 1989*)

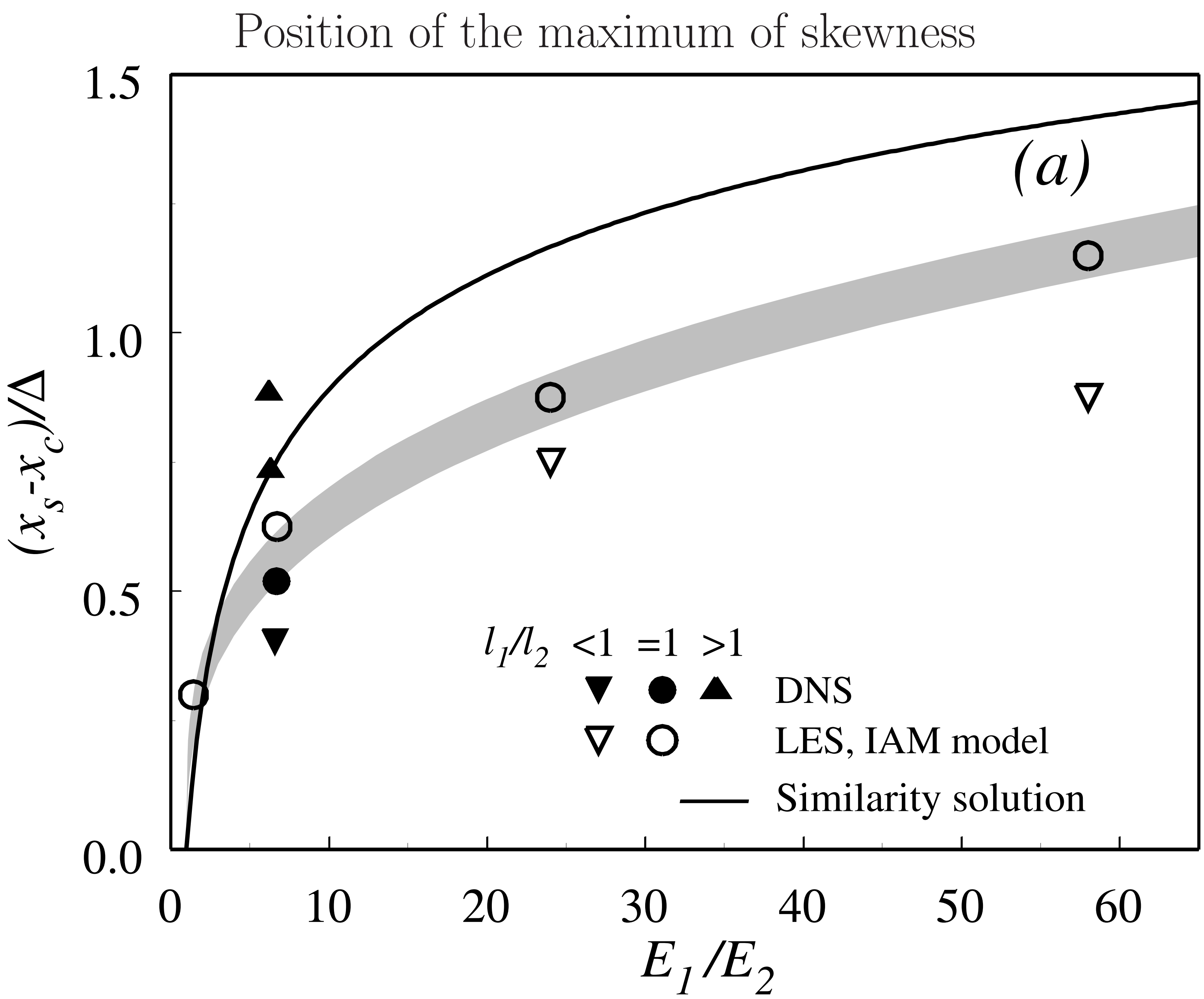
$$\varphi_{uu} = \frac{1+\mathcal{E}^{-1}}{2} - \frac{1-\mathcal{E}^{-1}}{2}\mathrm{erf}(\eta) \tag{18}$$

so that we can analitically carry out integrations to compute the velocity skewness

$$S = \frac{1-\mathcal{E}^{-1}}{\sqrt{\pi}}\mathrm{e}^{-\eta^2}\left[\frac{f}{4(1-2\alpha)}\left(1-\frac{4\nu}{A_1 f^2}\right)+2\beta(1-2\eta)\right]\times \left[\frac{1+\mathcal{E}^{-1}}{2} - \frac{1-\mathcal{E}^{-1}}{2}\mathrm{erf}(\eta)\right]^{-\frac{3}{2}} \tag{19}$$

Normalized energy and skewness distributions; $\mathcal{E} = 6.7$ and $\mathcal{L} = 1$.





Position of the maximum of skewness
(a)
$(x_s - x_c)/\Delta$
$E_1/E_2$
1.5
1.0
0.5
0.0
0
10
20
30
40
50
60
$l_1/l_2$ <1 =1 >1
DNS
LES, IAM model
Similarity solution

## Conclusions

The intermediate asymptotics of turbulence mixings in the absence of the production of turbulent kinetic energy has been considered.

- A similarity stage of the decay of shearless turbulence mixing always exist, even if it still depends on initial flow parameters.
- If $\mathcal{E}$ is far from unity, the mixing is very intermittent.
- intermittency smootly varies when passing through $\mathcal{L} = 1$:
  it increases when $\mathcal{L} > 1$, it is reduced when $\mathcal{L} < 1$
- when $\mathcal{L} = 1$, the intermittency increases with the energy ratio $\mathcal{E}$ with a scaling exponent that is almost equal to 0.29.
- a similarity decay of shearless mixing is consistent with single-point correlation equations