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Numerical experiments on the intermediate asymptotics of shear-free turbulent transport and diffusion. Associated similarity analysis.

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– D.Tordella, M.Iovieno 2006 "Numerical experiments on the intermediate asymptotics of shear-free turbulent diffusion", *Journal of Fluid Mechanics*, to appear.

– D.Tordella, M.Iovieno "Turbulence transport in the presence of a macroscale gradient", *APS, 58th Conference of the Division of Fluid Dynamicsd*, Chicago, Nov. 20-22, 2005.

– D.Tordella, M.Iovieno "The dependance on the energy ratio of the shear-free interaction between two isotropic turbulence" *Direct and Large Eddy Simulation 6 - ERCOFTAC Workshop*, Poitiers, Sept 12-14, 2005.

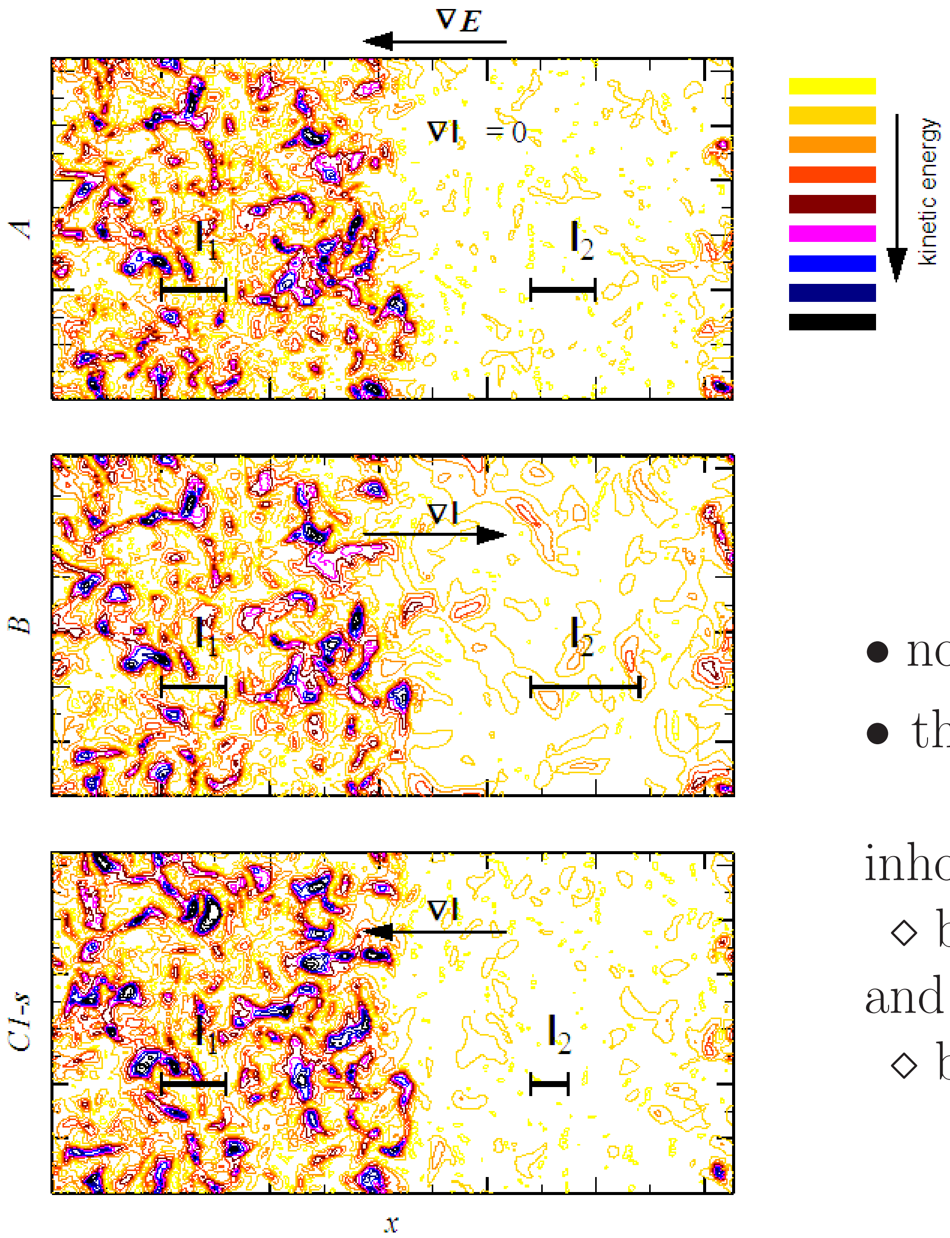
– D.Tordella, M.Iovieno "Self-similarity of the turbulence mixing with a constant in time macroscale gradient" *22nd IFIP TC 7 Conference on System Modeling and Optimization*, Torino, July 18-22, 2005.

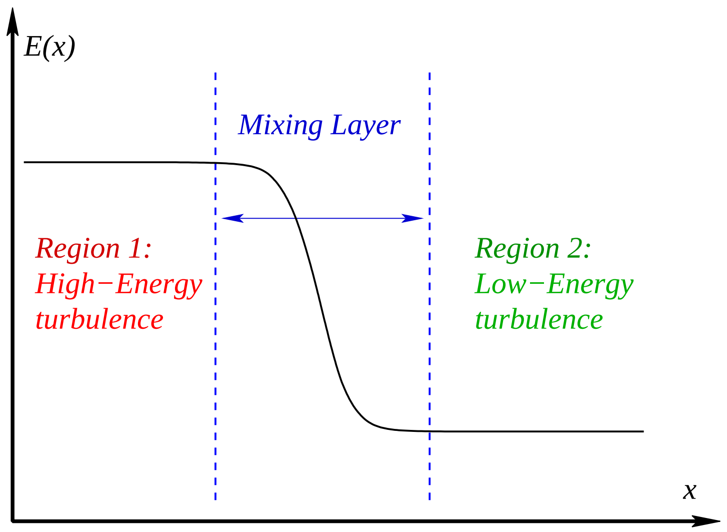
– M.Iovieno, D.Tordella 2002 "The angular momentum for a finite element of a fluid: A new representation and application to turbulent modeling", *Physics of Fluids*, **14**(8), 2673–2682.

– M.Iovieno, C.Cavazzoni, D.Tordella 2001 "A new technique for a parallel dealiased pseudospectral Navier-Stokes code." *Computer Physics Communications*, **141**, 365–374.

– M.Iovieno, D.Tordella 1999 "Shearless turbulence mixings by means of the angular momentum large eddy model", *American Physical Society - 52thDFD Annual Meeting.*

Shearless turbulence mixing.





- no mean shear $\Rightarrow$ *no turbulence production*
- the mixing layer is generated by the turbulence

inhomogeneity, i.e.:

◇ by the gradient of *turbulent energy*

and

◇ by the gradient of *integral scale*

Previous investigations:

Esperiments with grid turbulence:

– Gilbert B. *J. Fluid Mech.* **100**, 349–365 (1980).

– Veeravalli S., Warhaft Z. *J. Fluid Mech.* **207**,191–229 (1989).

Numerical simulations (DNS):

– Briggs D.A., Ferziger J.H., Koseff J.R., Monismith S.G. *J. Fluid Mech.* **310**, 215–241 (1996).

– Knaepen B., Debliquy O., Carati D. *J. Fluid Mech.* **414**, 153–172 (2004).

- in (passive) grid turbulence the higher energy is always associated to larger integral scales, so the two parameters are not independent ⇒ *guess about no intermittency in the absence of scale gradient and turbulence production*.
- numerical simulations reproduced the 3,3:1 laboratory experiment by Veeravalli and Warhaft.

New decay properties

- the two parameters, the *turbulent kinetic energy* ratio $\mathcal{E}$ and the *integral scale* ratio $\mathcal{L}$, has been independently varied
- the persistency of intermittency in the limit of no scale gradient ($\mathcal{L} \rightarrow 1$) and absence of turbulence production has been investigated.

In particular we present:

- *Part 1:* results from numerical simulations (DNS and LES, 2005 JFM, to appear)
- *Part 2:* intermediate asymptotics analysis ($\mathcal{L} \rightarrow 1$, 2005 IFIP TC7 and DLES6; $\mathcal{L} \neq 1$, in preparation)

Part 1: numerical experiments

Numerical simulations (DNS and LES) have been carried out with

- Fixed energy ratio $\mathcal{E} \sim 6.7$ and varying scale ratio $0.38 \leq \mathcal{L} \leq 2.7$
- No scale gradient ($\mathcal{L} = 1$) and variable energy ratio $1 \leq \mathcal{E} \leq 58.3$

- Reynolds number: $Re_\lambda \approx 45$ (DNS, LES) and $Re_\lambda \approx 450$ (LES, IAM model, Tordella & Iovieno *Phys.Fluids* 2002)

- Numerical method: Fourier-Galerkin pseudospectral on a $2\pi \times 2\pi \times 4\pi$ parallelepiped (Iovieno et al. *Comp.Phys.Comm.* 2001)
  Resolution: DNS $= 128^2 \times 256$, LES $= 32^2 \times 64$
- Initial conditions: two turbulent fields coming from simulations of decaying homogeneous isotropic turbulence.

Decay exponents

- The two homogeneous fields decay algebrically in time, according to theoretical (and experimental) results (see Karman and Howarth 1938, Sedov 1944, Batchelor 1953, Speziale 1995)

$$E = A(t + t_0)^{-n}$$

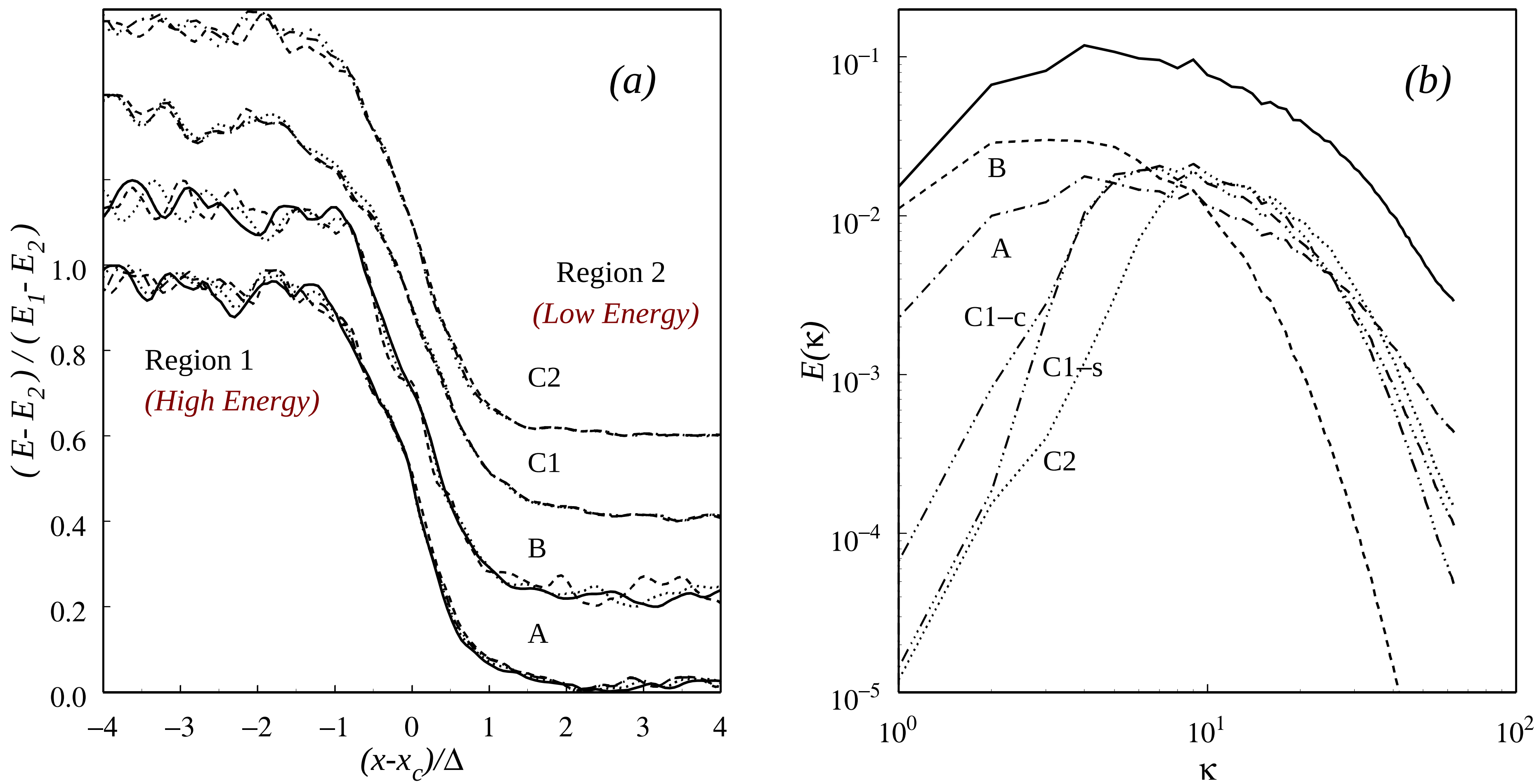
- Decay rates n_1, n_2 are higher than the limit, $n = 1$, for high Reynolds number, but still close to this value ($n_1 \approx n_2 \approx 1.2 - 1.4$), so that the energy and scale ratios remain nearly constant (up to 10%) during the decay

$$\frac{\mathcal{L}(t)}{\mathcal{L}(0)} = \left(1 + \frac{t}{t_{01}}\right)^{1 - \frac{n_1}{2}} \left(1 + \frac{t}{t_{02}}\right)^{-1 + \frac{n_2}{2}}$$

$$\frac{\mathcal{E}(t)}{\mathcal{E}(0)} = \left(1 + \frac{t}{t_{02}}\right)^{n_2} \left(1 + \frac{t}{t_{01}}\right)^{-n_1}$$

- All mixings have an intermediate self-similar stage of decay

Energy similarity profiles

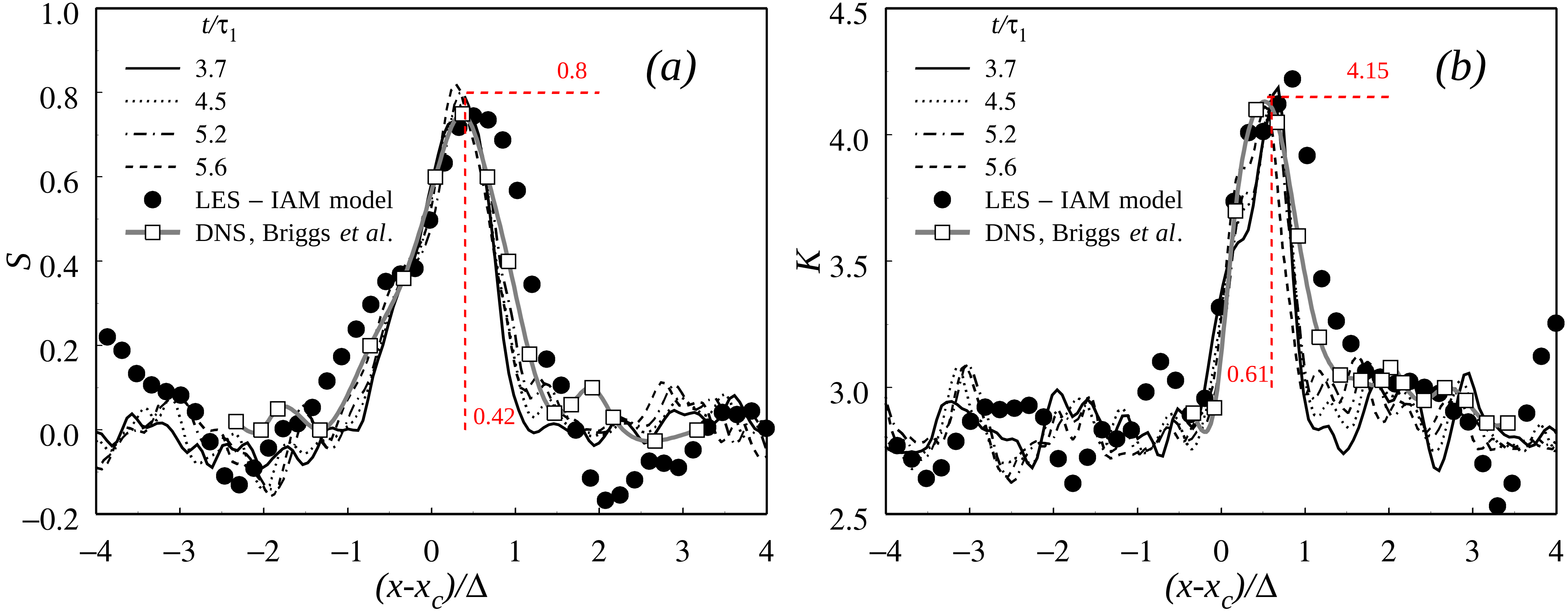


$\Delta(t)$ = mixing layer thickness, $\ell(t) = \frac{1}{3}\Sigma_i \frac{\int_0^\infty R_{ii}(r,t)dr}{R_{ii}(0,t)}$, where R_{ii} is the longitudinal velocity correlation (see e.g. Batchelor, 1953).

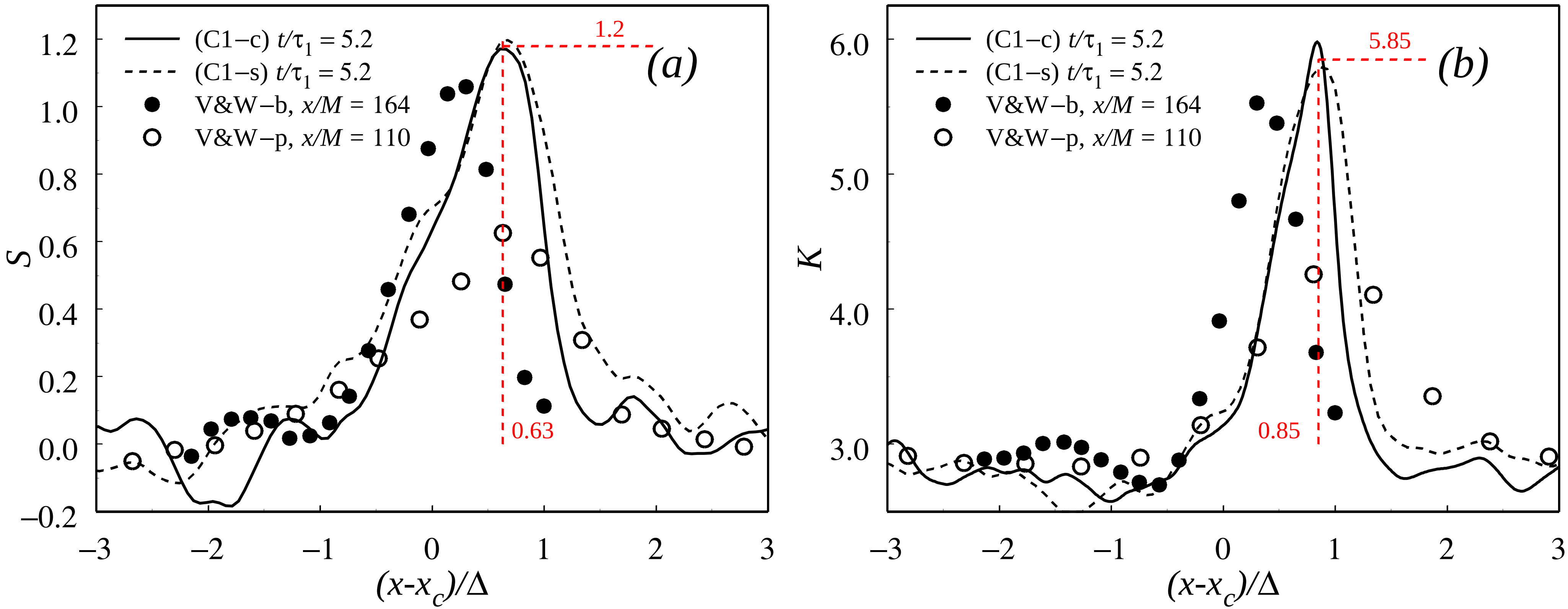
Higher order moments: skewness and kurtosis profiles

$$S = \frac{\overline{u^3}}{\overline{u^2}^{\frac{3}{2}}} \quad K = \frac{\overline{u^4}}{\overline{u^2}^2} \quad \Rightarrow \quad S \approx 0, \quad K \approx 3 \text{ in homogeneous isotropic turb.}$$

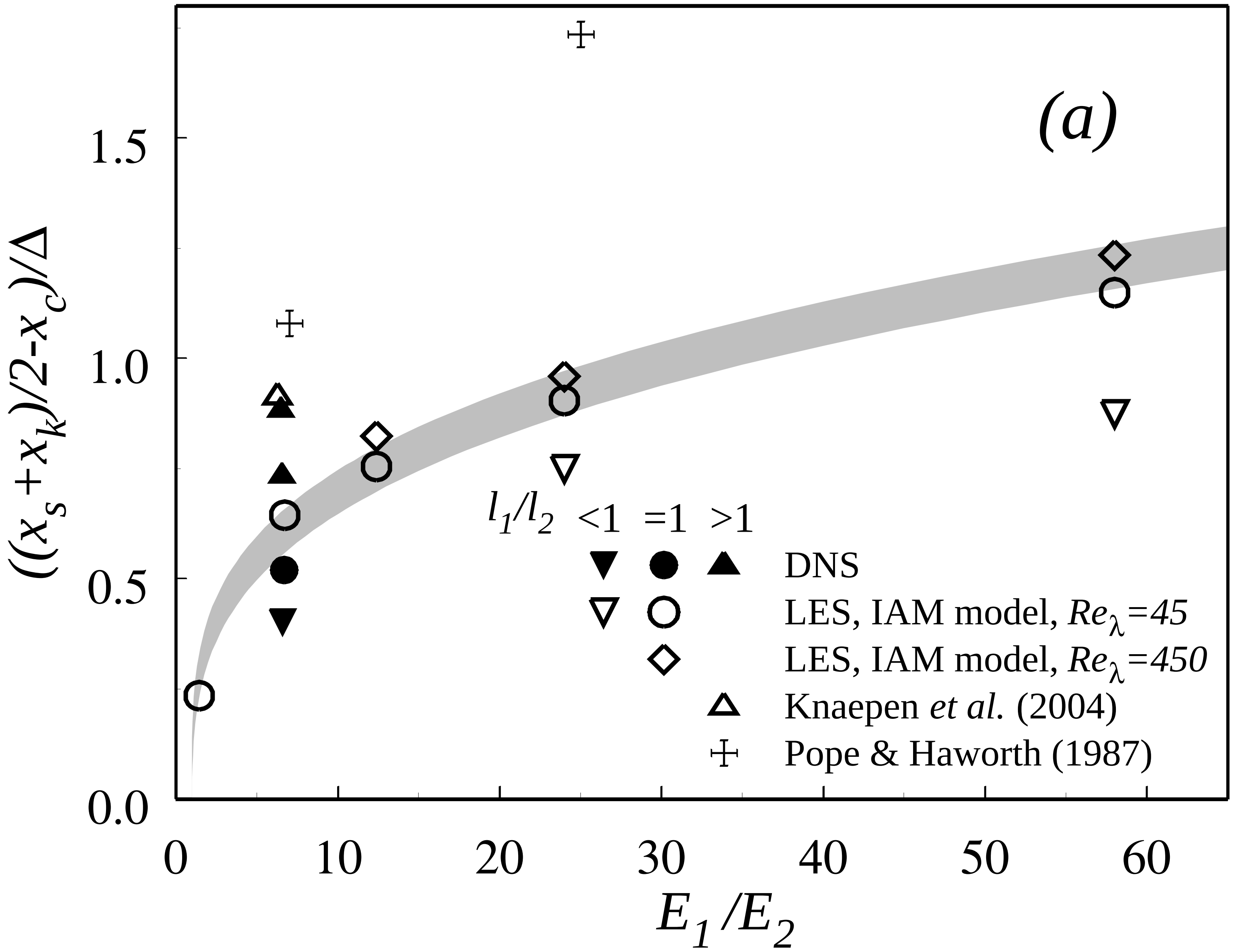
Case A: $E = 6.7,\ \mathcal{L} = 1$, the two fields have the same integral scale.

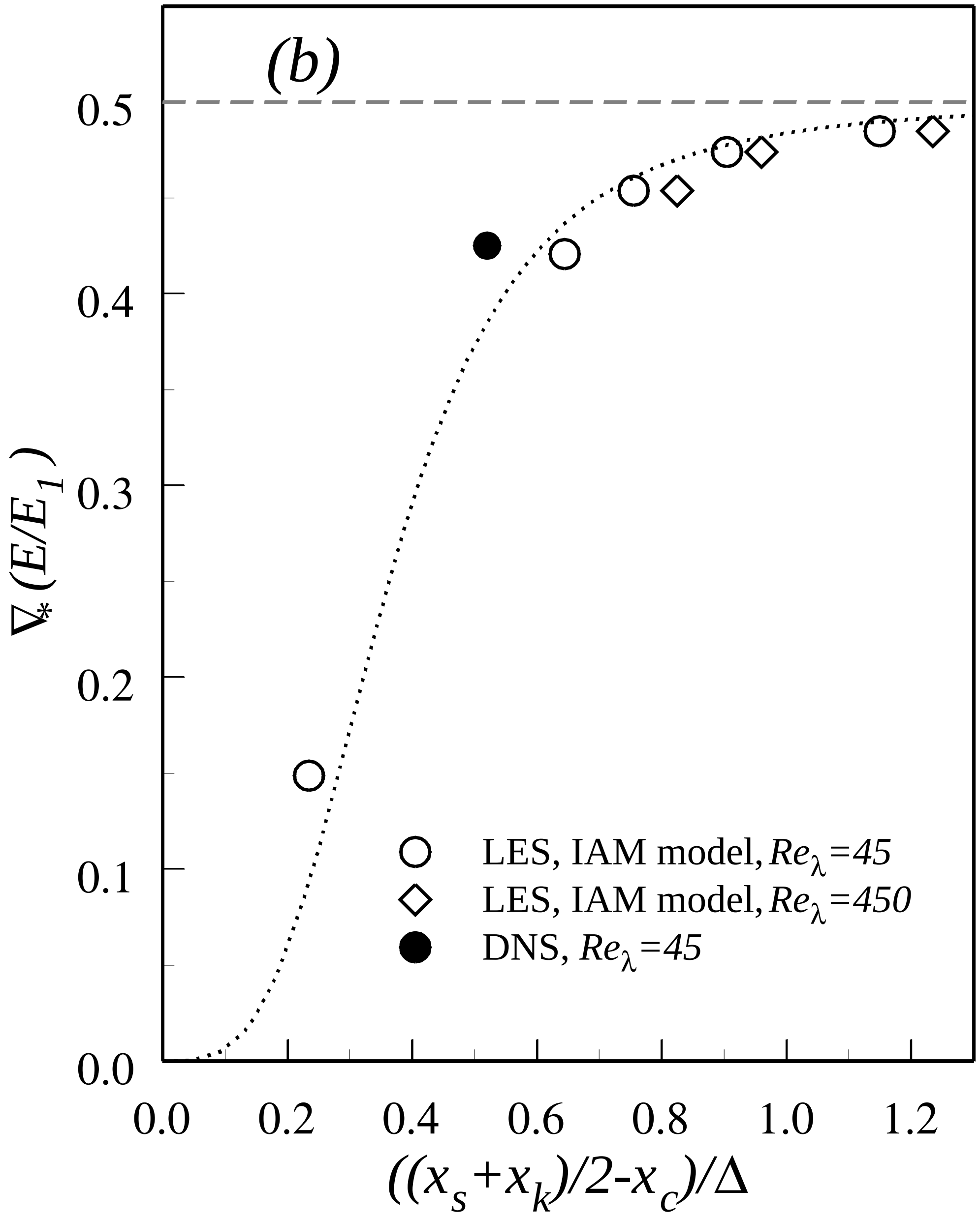


Case C: $\mathcal{E} = 6.5, \mathcal{L} = 1.5$: the gradients of energy and scales have the same sign: larger scale turbulence has more energy



Penetration - position of the maximum of skewness/kurtosis





Penetration with $\mathcal{L} = 1$

Scaling law (energy ratio):

$$\frac{\eta_s + \eta_k}{2} \sim a \left(\frac{E_1}{E_2} - 1 \right)^b$$

$$a \simeq 0.36, \quad b \simeq 0.298$$

Scaling law (energy gradient):

$$\nabla_*(E/E_1) \simeq (1 - \mathcal{E}^{-1})/2$$

$$\frac{\eta_s + \eta_k}{2} \sim a \left(\frac{2\nabla_*(E/E_1)}{1 - 2\nabla_*(E/E_1)} \right)^b$$

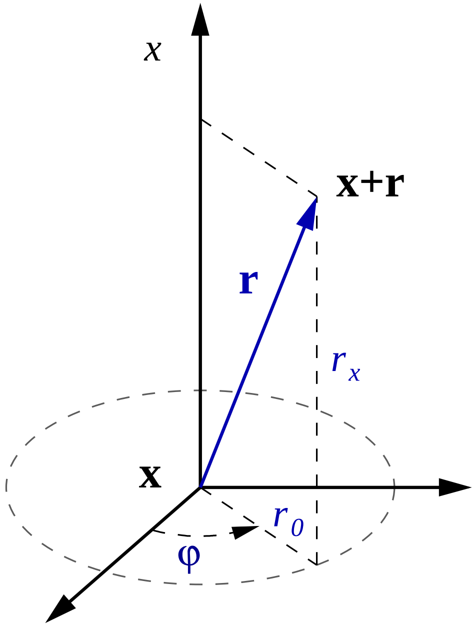
Part 2: similarity analysis

Properties of the numerical solutions:

- **A self-similar decay is always reached**
- **It is characterized by a strong intermittent penetration, which depends on the two mixing parameters:**
  - **the turbulent energy gradient**
  - **the integral scale gradient**

This behaviour must be contained in the turbulent motion equations:

- **the two-point correlation equation which allows us to consider both the macroscale and energy gradient parameters** $(B_{ij}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)})$**;**
- **the one-point correlation equation, the limit r → 0, which allows us to obtain the third order moment (skewness) distribution.**



Definition of two-point double correlation:

$$B_{ij}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)} \quad (1)$$

$$B_{pi}(\mathbf{x},\mathbf{r},t) = \overline{p(\mathbf{x},t)u_i(\mathbf{x}+\mathbf{r},t)} \quad (2)$$

$$B_{ip}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)p(\mathbf{x}+\mathbf{r},t)} \quad (3)$$

Definition of two-point triple correlation:

$$B_{ij|k}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x},t)u_k(\mathbf{x}+\mathbf{r},t)} \quad (4)$$

$$B_{i|jk}(\mathbf{x},\mathbf{r},t) = \overline{u_i(\mathbf{x},t)u_j(\mathbf{x}+\mathbf{r},t)u_k(\mathbf{x}+\mathbf{r},t)} \quad (5)$$

We consider the equation for the two-point lateral correlations in the limit $r_x \to 0$ (cylindrical polar coordinates)

$$\frac{\partial}{\partial t}B_{xx} + 2\frac{\partial}{\partial x}B_{xx|x} - 2\left(\frac{\partial B_{rx|x}}{\partial r_0} + \frac{B_{rx|x}}{r_0} + \frac{\partial B_{xx|x}}{\partial r_x}\right) =$$

$$= -2\frac{\partial}{\partial x}B_{px} + 2\frac{\partial}{\partial r_x}B_{px} +$$

$$+\nu\left[\frac{\partial^2}{\partial x^2} + 2\left(\frac{\partial^2}{\partial r_0^2} + \frac{1}{r_0}\frac{\partial}{\partial r_0} + \frac{\partial^2}{\partial r_x^2} - \frac{\partial^2}{\partial x \partial r_x}\right)\right]B_{xx} \qquad (6)$$

Hypothesis and semplifications

- **The two homogenous turbulences decay in the same way, thus**

$$E_1(t) = A_1(t+t_0)^{-n_1}, \quad E_2(t) = A_2(t+t_0)^{-n_2}$$

the exponents n_1, n_2 are close each other (numerical experiments, Tordella & Iovieno, 2005). Here, we suppose $n_1 = n_2 = n = 1$, a value which corresponds to $R_\lambda \gg 1$ (Batchelor & Townsend, 1948).

- **In the absence of energy production, the pressure-velocity correlation has been shown to be approximately proportional to the convective fluctuation transport (Yoshizawa, 1982, 2002)**

$$-\rho^{-1}\overline{pu} = a\,\frac{\overline{u^3} + 2\overline{v_1^2 u}}{2}, \quad a \approx 0.10,$$

- **Single-point second order moments are almost isotropic through the mixing:**

$$\overline{u^2} \simeq \overline{v_i^2}$$

Similarity hypothesis

The moment distributions are determined by

- **the coordinates x, r_0, t**
- **the energies $E_1(t)$, $E_2(t)$**
- **the scales $\ell_1(t)$, $\ell_2(t)$.**

We introduce the variable separation

$$B_{xx}(x, r_0, t) = B_{xx}(-\infty, 0, t)\varphi_{xx}(\eta, \xi) \quad (7)$$

$$B_{xx|x}(x, r_0, t) = B_{xx}^{\frac{3}{2}}(-\infty, 0, t)\varphi_{xx|x}(\eta, \xi) \quad (8)$$

where

$$\eta = \frac{x}{\Delta(t)}, \qquad \xi = \frac{r_0}{\ell(x, t)}$$

and where $\Delta(t)$ is the mixing thickness and $\ell(x, t)$ is the local integral scale. $B_{xx}(-\infty, 0, t)$ is the one-point correlation in the homogeneous region of high kinetic energy, which is equal to $(2/3)E_1(t)$.

$\Rightarrow$ similarity conditions:

By introducing the similarity relations in the equation and by imposing that all the coefficients must be independent from x, t, it is obtained

$$\Delta(t) \propto \ell_1(t)$$

and by taking the limit $\xi \to 0 \Rightarrow$ similarity equation:

$$\frac{\partial \varphi_{xx|x}}{\partial \eta} = \frac{2f(R_{\lambda_1})}{3}\left\{\frac{1}{2}\eta\frac{\partial \varphi_{xx}}{\partial \eta} + \frac{3}{2f(R_{\lambda_1})}\frac{1}{R_{\ell_1}}\frac{\partial^2 \varphi_{xx}}{\partial \eta^2} + \left[\varphi_{xx} + \frac{3}{2f(R_{\lambda_1})}\frac{1}{R_{\ell_1}}\frac{1}{\lambda_I^2}\frac{\partial^2 \varphi_{xx}}{\partial \xi^2}\right]\right\} \tag{9}$$

with boundary conditions

$$\lim_{\eta \to -\infty} \varphi_{xx}(\eta) = \frac{2}{3}, \quad \lim_{\eta \to +\infty} \varphi_{xx}(\eta) = \frac{2}{3}\,\mathcal{E}^{-1}, \quad \lim_{\eta \to \pm\infty} \varphi_{xxx}(\eta) = 0$$

By introducing a Taylor microscale and an integral scale defined on the lateral double velocity correlation

$$\frac{1}{\lambda^2} = -\frac{B''_{NN}(0)}{2B_{NN}(0)} \tag{10}$$

$$\ell = 2\int_0^\infty \frac{B_{NN}(r)}{B_{NN}(0)}\mathrm{d}r \tag{11}$$

By recalling

$$\ell(x,t) = \ell_1(t)\lambda_I(\eta)$$

and representing the Taylor microscale as

$$\lambda(x,t) = \ell_1(t)\lambda_T(\eta).$$

it is possible to write[1]

$$2\int_0^\infty \frac{\varphi_{xx}(\eta,\xi)}{\varphi_{xx}(\eta,0)}\mathrm{d}\xi = 1$$

$$\frac{1}{\lambda_T^2} = -\frac{1}{2\lambda_I^2}\frac{1}{\varphi_{xx}(\eta,0)}\frac{\partial^2\varphi_{zz}}{\partial\xi^2}(\eta,0)$$

Thus

$$\frac{\partial^2\varphi_{xx}}{\partial\xi^2}(\eta,0) = -2\frac{\lambda_I^2(\eta)}{\lambda_T^2(\eta)}\varphi_{xx}(\eta,0)$$

[1] the first is just a normalization condition, which is implied by the position $\xi = r_0/\ell(x,t)$.

The previous similarity equation may then be written as

$$\frac{\partial \varphi_{xx|x}}{\partial \eta} = \frac{2f(R_{\lambda_1})}{3}\left\{\frac{1}{2}\eta\frac{\partial \varphi_{xx}}{\partial \eta} + \frac{3}{2f(R_{\lambda_1})}\frac{1}{R_{\ell_1}}\frac{\partial^2 \varphi_{xx}}{\partial \eta^2} + \varphi_{xx}\left[1 - \frac{3}{f(R_{\lambda_1})}\frac{1}{R_\ell(\eta)}\frac{\lambda_I^2(\eta)}{\lambda_T^2(\eta)}\right]\right\} \tag{12}$$

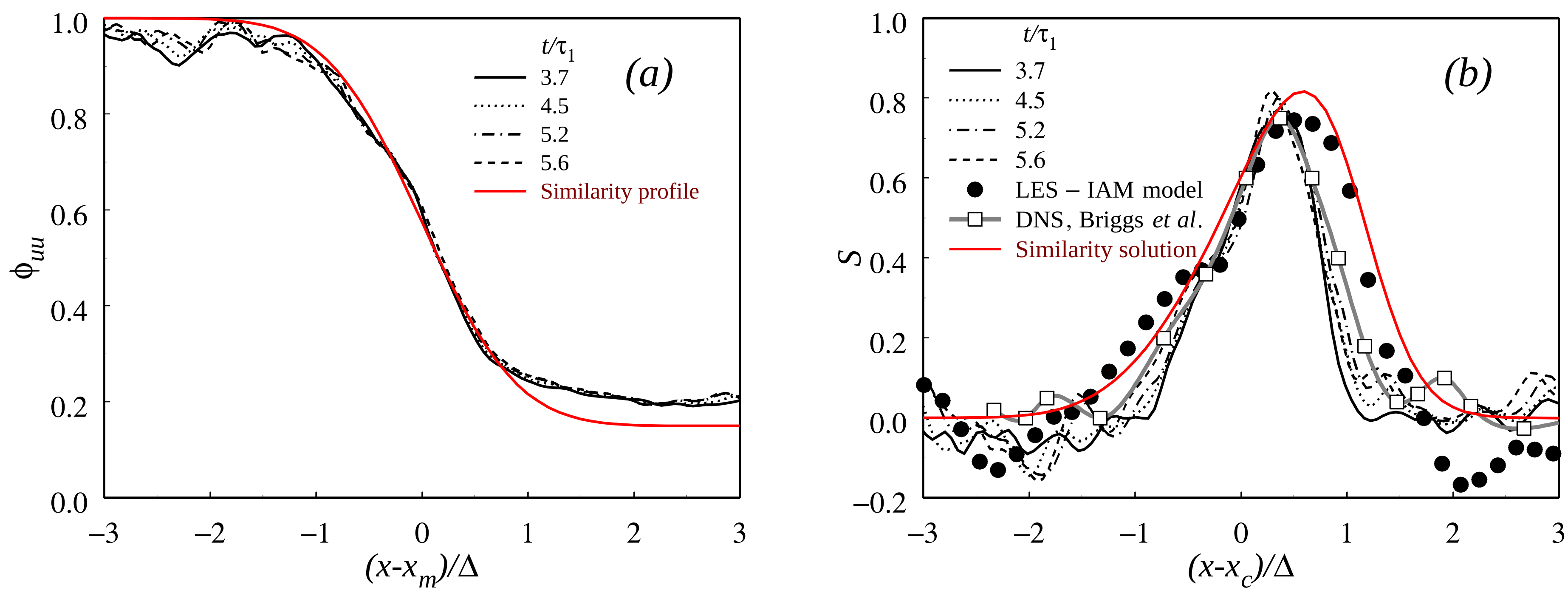
In case the ratio λ_T/λ_I is constant, then the term inside square brackets will also be constant. But this term vanishes when $\eta \to \pm\infty$, which means that it is always zero. So that

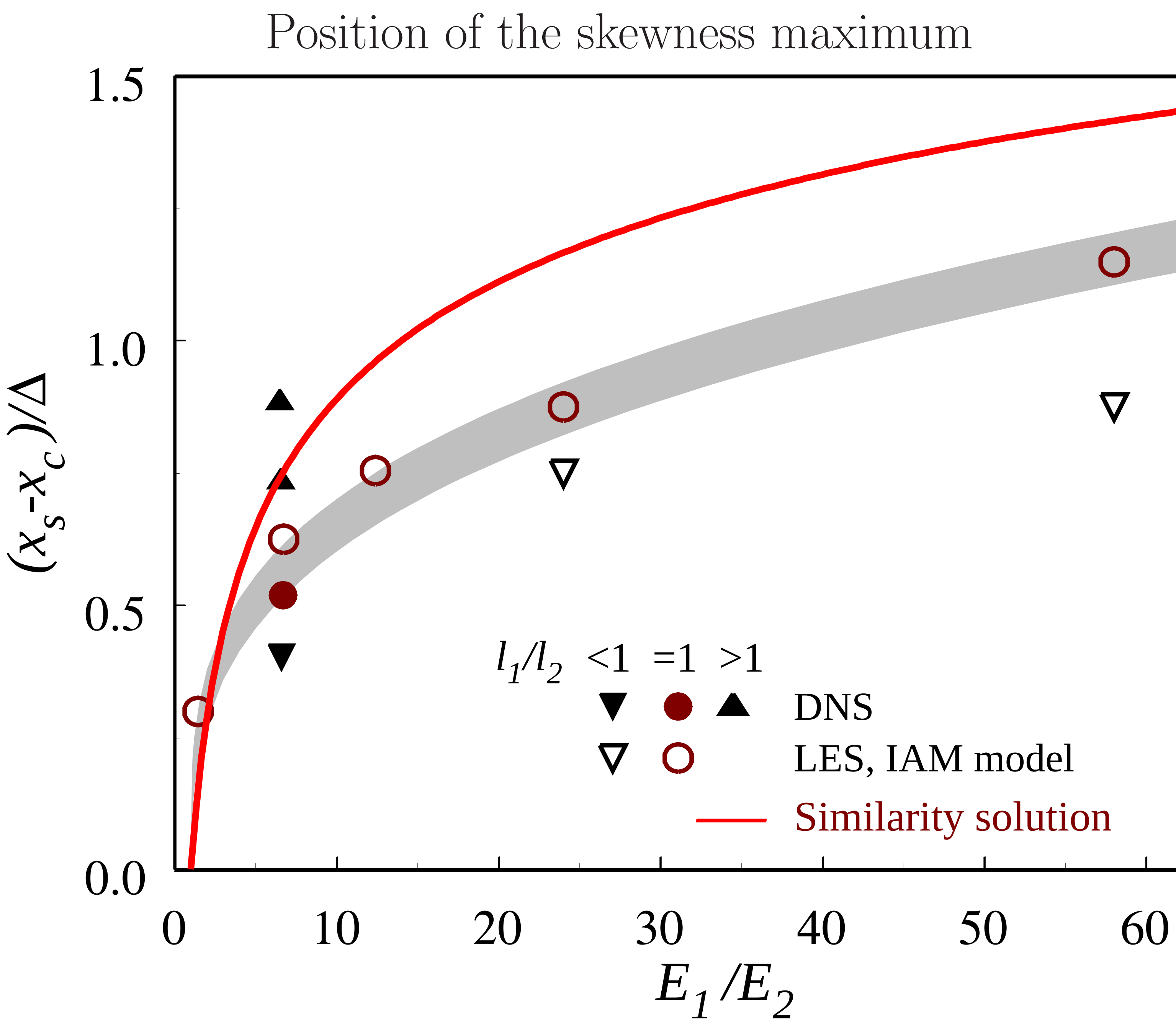
$$\frac{\lambda_I^2(\eta)}{\lambda_T^2(\eta)} = \frac{2f(R_{\lambda_1})}{3}R_\ell(\eta) \propto R_\ell(\eta) \tag{13}$$

and the solution is independent on the scale variation.

We take this position as a representation of the mixing with $\mathcal{L} = \ell_1(t)/\ell_2(t) = 1$ (where subscripts 1 and 2 refer to the high/low energy regions respectively)

Normalized energy and skewness distributions; $\mathcal{E} = 6.7$ and $\mathcal{L} = 1$.





Position of the skewness maximum
1.5
1.0
0.5
0.0
0
10
20
30
40
50
60
$(x_s-x_c)/\Delta$
E_1/E_2
l_1/l_2 <1 =1 >1
DNS
LES, IAM model
Similarity solution

When $\mathcal{L} \neq 1$ the ratio λ_T/λ_I cannot be constant inside the mixing, which implies that the shape of the double correlation, even if normalized with the local energy and integral scale, is changing through the layer.
We can suppose

$$\lambda_I(\eta) = \frac{1+\mathcal{L}^{-1}}{2} - \frac{1-\mathcal{L}^{-1}}{2}F(a\eta) \tag{14}$$

$$\lambda_T(\eta) = \left(\frac{3}{f(R_{\lambda_1})R_\ell(\eta)}\right)^{\frac{1}{2}} \left\{\frac{1+\mathcal{L}^{-1}}{2} - \frac{1-\mathcal{L}^{-1}}{2}F(a\eta)\right\} \left(1 - bF^{(k)}(a\eta)\right)^{-\frac{1}{2}} \tag{15}$$

The two parameter a e b are function of $\mathcal{L}$, $a \neq 1$ makes the distribution of integral scale different from the energy distribution (modifies the thickness of "scale mixing layer" with respect to that of the energy), $b \neq 0$ modifies the shape of the correlation function inside the mixing (changes the distribution of the Taylor microscale with respect to that of the integral scale). We obtaine:

$$\left[1 - \frac{3}{f(R_{\lambda_1})}\frac{1}{R_\ell(\eta)}\frac{\lambda_I^2(\eta)}{\lambda_T^2(\eta)}\right] = -\frac{b}{a^k}\frac{2}{1-\mathcal{L}^{-1}}\lambda_I^{(k)}(\eta).$$

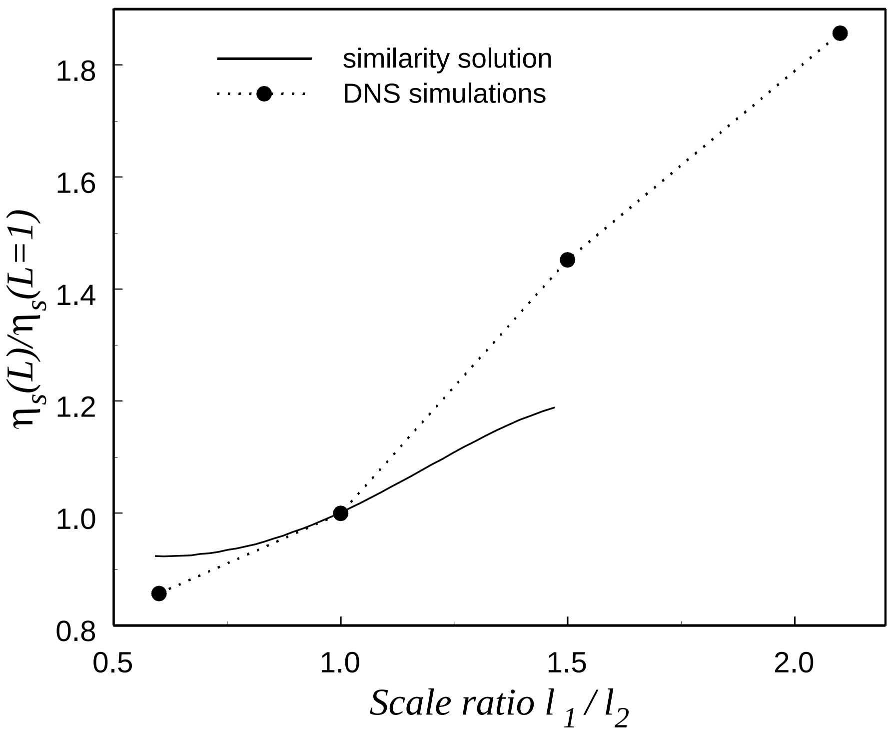
The integral of this term in $\eta = (-\infty, \infty)$ vanishes for $\forall k \geq 2$.

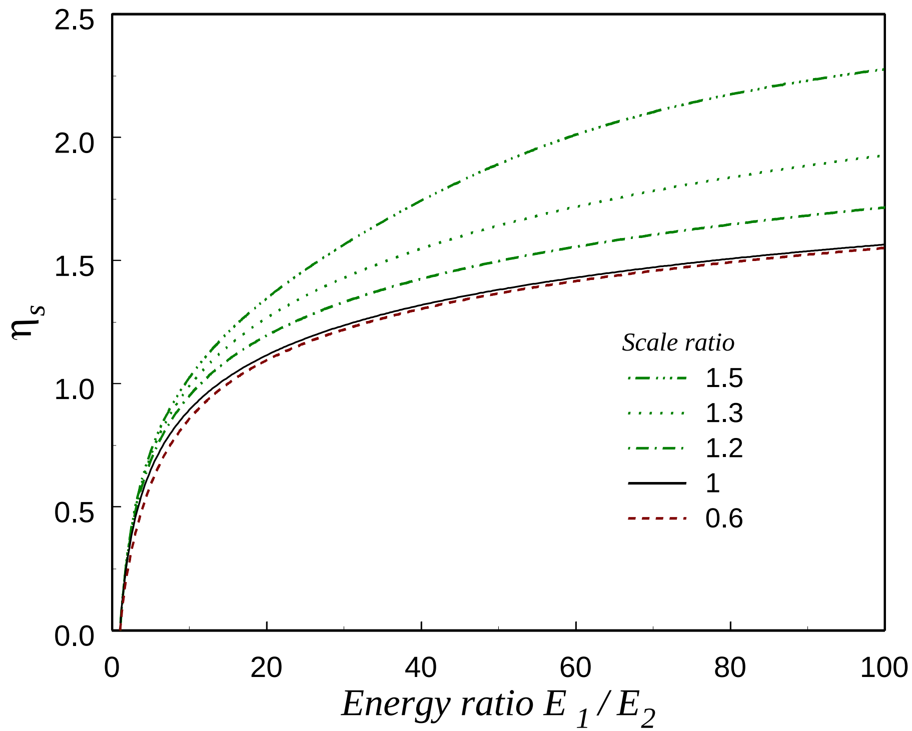
The associated contribution to the skewness is an additive term

$$S = \ldots + -\frac{b}{a^k}\frac{2}{1-\mathcal{L}^{-1}}\,\lambda_I^{(k)}(\eta)\left[\frac{1+\mathcal{E}^{-1}}{2} - \frac{1-\mathcal{E}^{-1}}{2}F(\eta)\right]^{-3/2}$$

For instance, by taking $k = 2$ **and** $b = 0.1$**,**

$$a = 3,4 - 2,4\mathcal{L}$$





CONCLUSIONS

The intermediate asymptotics of the turbulence diffusion in the absence of production of turbulent kinetic energy are considered.

- **An intermediate similarity stage of decay always exists.**
- **When the energy ratio $\mathcal{E}$ is far from unity, the mixing is very intermittent.**
- **when $\mathcal{L} = 1$, the intermittency increases with the energy ratio $\mathcal{E}$ with a scaling exponent that is almost equal to 0.29.**
- **intermittency smoothly varies when passing through $\mathcal{L} = 1$:**
  it increases when $\mathcal{L} > 1$ (*concordant* gradient of energy and scale),
  it is reduced when $\mathcal{L} < 1$ (*opposite* gradient of energy and scale)

indipendently from the numerical simulations

- **the self-similar decay of the shearless mixing is consistent with the similarity analysis of the two-point double velocity correlation equation.**
- **a relation between the integral scale and the Taylor microscale variation across the mixing has been obtained.**

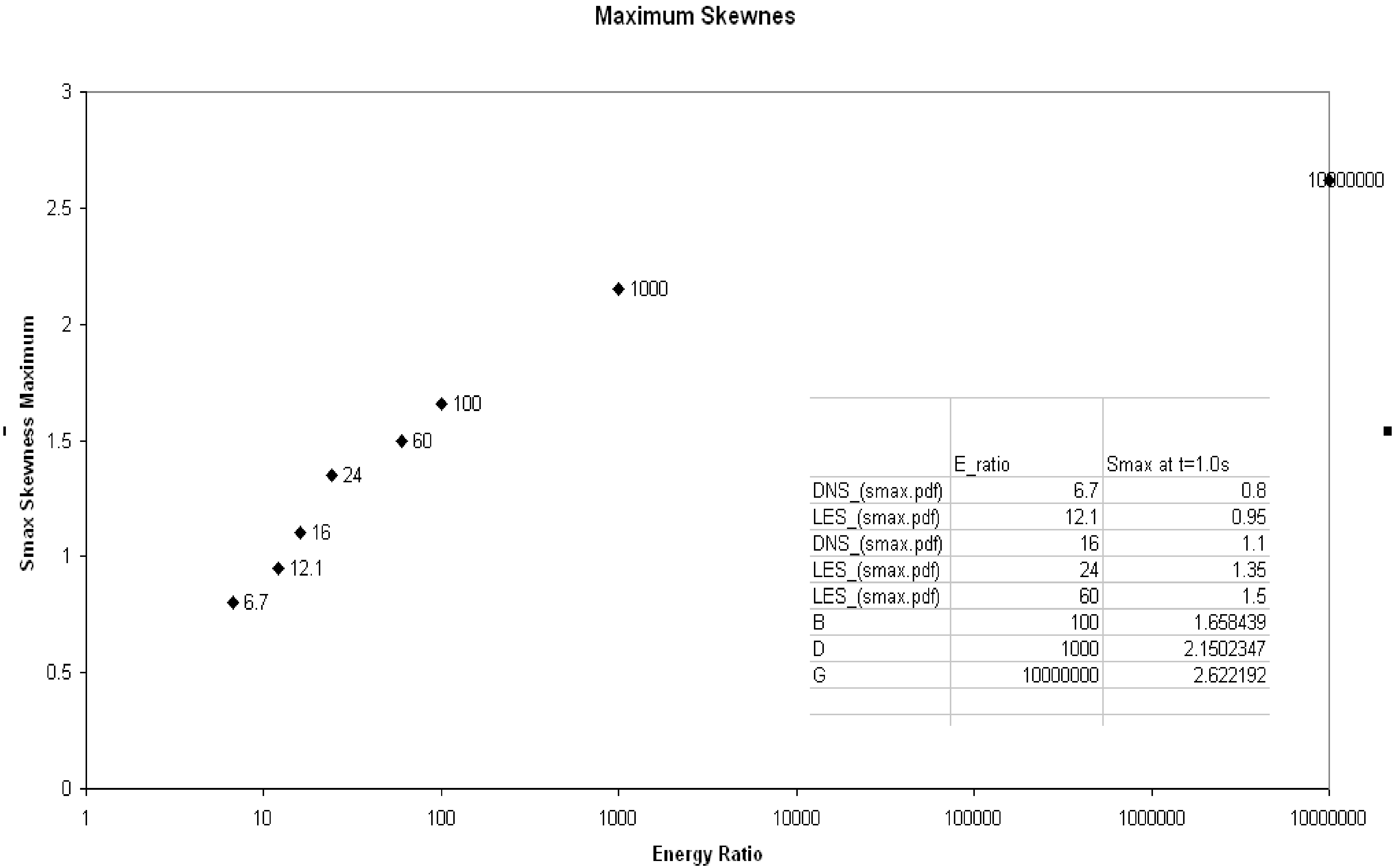
$$-\varphi_{zz} + \frac{1}{2}\left[-\eta\frac{\partial \varphi_{zz}}{\partial \eta} + \xi\eta\lambda_I'\frac{\partial \varphi_{zz}}{\partial \xi}\right] + \frac{3}{2f(R_{\lambda_1})}\left[\frac{\partial \varphi_{zz|z}}{\partial \eta} - \xi\lambda_I'\frac{\partial \varphi_{zz|z}}{\partial \xi}\right] + \frac{3}{2f(R_{\lambda_1})}\frac{1}{\lambda_I}\frac{\partial \varphi_{zz|z}}{\partial \xi} =$$

$$= \frac{3}{2f(R_{\lambda_1})}\frac{1}{R_{\ell_1}}\left\{\left[\frac{\partial^2 \varphi_{zz}}{\partial \eta^2} - \xi\lambda_I'\left(\frac{\partial^2 \varphi_{zz}}{\partial \eta\partial \xi} - \lambda_I'\frac{\partial \varphi_{zz}}{\partial \xi} - \xi\lambda_I'\frac{\partial^2 \varphi_{zz}}{\partial \xi^2}\right) - \xi\lambda_I''\frac{\partial \varphi_{zz}}{\partial \xi}\right]\right.$$

$$\left. + \left[-\frac{\lambda_I'}{\lambda_I^2}\frac{\partial \varphi_{zz}}{\partial \xi} + \frac{1}{\lambda_I}\frac{\partial^2 \varphi_{zz}}{\partial \eta\partial \xi}\right] + \frac{1}{\lambda_I^2}\frac{\partial^2 \varphi_{zz}}{\partial \xi^2}\right\} \quad (16)$$

$$\frac{\partial \varphi_{zz|z}}{\partial \eta} = \frac{2f(R_{\lambda_1})}{3}\left\{\frac{1}{2}\eta\frac{\partial \varphi_{zz}}{\partial \eta} + \frac{3}{2f(R_{\lambda_1})}\frac{1}{R_{\ell_1}}\frac{\partial^2 \varphi_{zz}}{\partial \eta^2} + \varphi_{zz}\left[1 - \frac{3}{2f(R_{\lambda_1})}\frac{1}{R_{\ell_1}}\frac{\lambda_I^2(\eta)}{\lambda_T^2(\eta)}\right]\right\} \quad (17)$$

Shearless turbulence mixing. Skewness maximum and $\mathcal{E}$



	E_ratio	Smax at t=1.0s
DNS_(smax.pdf)	6.7	0.8
LES_(smax.pdf)	12.1	0.95
DNS_(smax.pdf)	16	1.1
LES_(smax.pdf)	24	1.35
LES_(smax.pdf)	60	1.5
B	100	1.658439
D	1000	2.1502347
G	10000000	2.622192

Shearless turbulence mixing. Skewness maximum and penetration